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ON MULTIFORM FUNCTIONS DEFINED BY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By Pierre Boutroux.

Addressing the third international mathematical congress, in 1904, Painlevé stated that:* Generally speaking, an irreducible differential equation should be regarded as integrated or solved if, by any analytical process, we are able to represent the general solution in its whole field of existence with any required approximation, the representation displaying the fundamental properties of the equation, showing how the initial conditions enter in, etc.

I wish to suggest a method which, from such a viewpoint, may be said to solve a number of differential equations of the first order. In addition, this method may be of some interest as introducing certain multiform functions (associated with the solutions of the differential equations) which have remarkable automorphic properties.

As the preliminary studies on which the method is based, and the discussion of the numerous cases which it involves, cannot be exposed without entering into long developments, it may not be useless to state the main results† separately and illustrate them by an example, in the brief summary which I am presenting here. A longer article on the same subject will be printed in the Annales de l'Ecole Normale Supérieure.

1. **Classification of a family of differential equations.** The only equation of the first order and degree which defines uniform functions is, as is well known, the Riccati equation. I have made repeated efforts to investigate another equation which, for many reasons, seems to be next in simplicity, that is

$$y' + A_0 + A_1 y + A_2 y^2 + A_3 y^3 = 0, \tag{1}$$

where the A's are *rational* functions of x, and, first of all, polynomials. The study of this equation, which I shall write hereafter

$$zz' = A_0 z^3 + A_1 z^2 + A_2 z + A_3; \qquad z = y^{-1}, \tag{2}$$

leads to following classification:

First Case. Let $A_0 = A_1 = 0$ identically. Then, in the neighborhood of $x = \infty$, the single *branches* of the solutions of (2) are of the *algebroid*

* Verhandl. des III. Intern. Math. Congr., Heidelberg, p. 96.

† Some of these results have been given in short papers published in the Comptes rendus de l'Académie des Sciences, Paris.

type. I mean by this that, if we follow *on all straight lines starting from any point* a solution $z(x)$ of (2) (thus obtaining a *branch* of this solution), there will exist an algebraic function $Z(x)$ such that $z(x)/Z(x)$ approaches 1 when x approaches ∞. (In other words the branch is asymptotic to an algebraic function).

Second Case. Let A_0 alone be identically equal to 0. Then, in the neighborhood of $x = \infty$, the branches $z(x)$ are of the *exponential type*; they behave like exponential functions; in particular, no one of them ever becomes infinite, but presents an infinite number of *roots* (which, for it, are singular points, since equation (2) offers a singularity wherever $z = 0$).

Third Case. A_0 and A_1 are *not* 0. Then the branches $z(x)$ are, in the vicinity of $x = \infty$, of the *meromorphic type*; for each of them we have an infinite number of roots and an infinite number of infinites.

In view of this, it will be natural to leave cases 2 and 3 for a later investigation and to concentrate our efforts on Case 1.

Now, in this case, calling m_2 and m_3 the degrees of A_2 and A_3 in x, I have been led to make further distinctions. I say that equation (3),

$$zz' = A_2z + A_3, \tag{3}$$

is of *type* A when $m_3 > 2m_2 + 1$,
of *type* B when $m_3 < 2m_2 + 1$,
of *type* C when $m_3 = 2m_2 + 1$.

The type which I am going to consider at present is *type* A.

2. **A simple equation of type A here proposed as an example.** To make as clear as possible the facts which I wish to submit, I shall limit my statements, in this summary, to one of the simplest examples of equations of type A which can be formed. This is an equation (3) where $m_3 = 3$, $m_2 = 0$, namely

$$zz' = 3mz + 2(x^3 - 1) \qquad (m \text{ constant}). \tag{4}$$

To discuss this equation fully, I shall assume later that m is a real negative number of small absolute value.

3. **First properties of the solutions of** (4). The solutions $z(x)$ offer no *poles*, nor other points where they are infinite. Their only *algebraic singular points* are their roots (points where $z = 0$), each of which exchanges two determinations of the vanishing branch. The only *transcendental* singular points of the equation are

$$x = \alpha = e^{2i\pi/3};\ x = \beta = e^{-(2i\pi/3)};\ x = \gamma = 1, \qquad \text{for special branches,}$$

$$x = \infty, \qquad \text{for all branches.}$$

Let us call $\bar{x}$ a point which we make approach ∞ on the negative

real axis; call $2y$ the value of $z^2 - \bar{x}^4 + 4\bar{x}$ at this point, and follow any path from $\bar{x}$ to any point x. If $z(x)$ is a solution of (4), we have in x:

$$(5) \qquad Z = z^2 = x^4 - 4x + 2y + 6m \int_{\infty}^{x} z dx,$$

the integral being taken along the appointed path from $\bar{x}$ (removed to ∞) to x.

Let us consider such branches $Z(x)$ which are obtained by starting from $\bar{x}$ (that is ∞) with a given y and following the set of all straight lines parallel to the real axis. Such a branch is, in the neighborhood of $x = \infty$, asymptotic to the polynomial $x^4 - 4x + 2y$; furthermore, if $|y|$ is large, the ratio of its value to $x^4 - 4x + 2y$ for any x on the straight lines described is close to 1 and approaches 1 when y approaches ∞. It follows that, if $|y|$ is large, the said branch offers exactly 4 roots which are respectively in the vicinities of the roots of $x^4 - 4x + 2y$. (This fact which is always true for any given equation (4) as soon as $|y|$ is above a certain numerical function of the coefficient m in the equation, would be true for all values of y if we gave to m a value approaching 0).

As $Z = z^2$, the branches $z(x)$ corresponding to the branches $Z(x)$ just described are asymptotic to $\sqrt{x^4 - 4x + 2y}$. There are, therefore, two sets of branches $z(x)$, the ones becoming infinite like $+ x^2$ (on the straight lines followed) and the others like $- x^2$.

Each of the branches Z or z so defined may be represented in the whole x-plane by a limited number of Taylor's series; that is obvious for a large $|y|$ from what has just been said, and it can be proved that it is always so. Let us add that there exist, in fact, other much more convenient single expansions for these functions (or rather branches); but I shall not enter into this question here, and will only point out that, as soon as a value of y is given, we are able to represent the corresponding branch wholly and to determine its behavior. The only problem left therefore, in order to know any one of the multiform functions in its whole field of existence is to determine the whole set of values of y which belong to the same function (that is, which define branches $z(x)$ all belonging to the same multiform function).

4. **Another definition of parameter** y. In order to understand the part played in the question by parameter y, it is useful to show its connection with another parameter which may be defined as follows:

In the neighborhood of $x = \infty$, as we have seen, there are two sets of branches $z(x)$. Let us consider those branches that become infinite like $+ x^2$. They may be represented by the expansion

(6) $z = x^2 + mx + m^2 +$ *expansion in powers of* x^{-1} and *of* $(C_1 + \eta_1 \log x)x^{-2}$,

where $\eta_1 = 6m(m^3 - 1)$, while C_1 is a parameter.

The set of branches becoming infinite like $-x^2$ may be represented by a similar expansion $z = -x^2 + mx + $ *terms in* x^{-1} *and* $(C_2 + \eta_2 \log x)x^{-2}$, where $\eta_2 = 6m(m^3 + 1)$ and C_2 is another parameter.

Let us decide always to give to $\log x$ in (6) *such values as have their imaginary part between* 0 *and* $2i\pi$: there is, then, a one-to-one correspondence between the different values of C_1 and the different branches z defined by (6). Similarly, we define a one-to-one correspondence between the values of C_2 and the branches of the second set.

According to these assumptions, there will correspond to any individual multiform function $z(x)$: (1) a set of values of C_1; (2) a set of values of C_2. We shall, hereafter, fix our attention on the C_1 only. It is obvious that whatever we are doing with C_1 could be done with C_2 also.

To introduce the parameter C_1 is, in fact, to give a new definition of y, for we have (taking the square of the right hand member of (6))

$$y = -2m + \frac{5}{2}m^4 + C. \tag{7}$$

5. **The group problem.** We may now state the question before us as follows:

Starting from $\bar{x}$ removed to ∞ with z equal to a branch of the first set (that is becoming infinite like $+x^2$ and corresponding to a value of C_1) let us follow z along any possible closed path which brings us back in $\bar{x}$ with z equal to a branch of the same set. If the two values of C_1 corresponding to the beginning and the end of the path are different, we may say that the path *exchanges* those two values of C_1 or the corresponding values of y (defined by (7)). In other words, the closed path operates a *substitution* (S) on y, say $[y_0, y_1]$. Our problem is to find all possible values y_1 which may be so exchanged with the same y_0. In other words, we have to determine all the substitutions (S).

It follows from formula (5), that if we call Γ any of the closed paths interchanging two values of y, the substitutions before us are the substitutions $[y,\ y + 3m\int_\Gamma z\,dx]$. And, knowing (see 3) that z is asymptotic to the square root of a polynomial of the fourth degree, we may foresee that the study of the substitution (S) will not be without analogy with that of elliptic integrals. The condition, for instance, that all values of y considered in the substitutions should correspond to values of C_1 is exactly the condition which is needed in order to define the periods of elliptic integrals: only such closed paths, namely, can define periods of the integral $\int \sqrt{x^4 - 4x + y}\,dx$, on which $\sqrt{x^4 - 4x + y}$ has the same sign at the initial and final point, that is to say such paths as effect an *even* number of single permutations.

The substitutions (S) satisfying the conditions prescribed evidently form a discontinuous group: it is this group—group (G)—that we have to investigate.

6. **Introducing the ψ-functions.** The chief question concerning group (G) is obviously the following: Is it possible to build up that group with a limited number of simple fundamental substitutions? To answer this question, we shall introduce new multiform functions, which I call ψ, and which, for the special equation which I have here in view, may be defined as follows:

If, in equation (5), we make $m = 0$ and $y = 0$, we have $z = \sqrt{x^4 - 4x}$. The function z then has only 4 roots (or critical points) which are $x_1 = 0$, $x_2 = e^{-(2i\pi/3)} \sqrt[3]{4}$, $x_3 = \sqrt[3]{4}$, $x_4 = e^{2i\pi/3} \sqrt[3]{4}$. From $\bar{x}$ (removed to ∞ in the direction of the negative real axis, see 3), we draw a closed negatively sensed circuit Γ_1 surrounding x_1 and x_2, and another closed positively

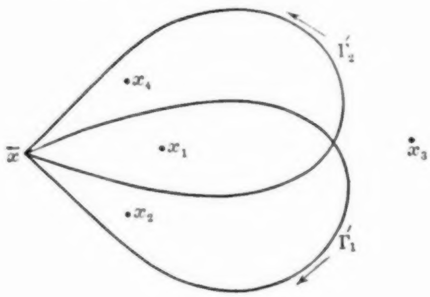

FIG. 1.

sensed circuit Γ_2 surrounding x_1 and x_2. For small values of y and m, the $z(x)$ defined by (5) (the one which in $\bar{x}$ becomes infinite like $+ \bar{x}^2$) will be holomorphic all along Γ_1 and Γ_2: therefore the equalities

$$(8) \qquad \psi_1(y) = y + 3m \int_{\Gamma_1} z d\bar{x}; \qquad \psi_2(y) = y + 3m \int_{\Gamma_2} z dx$$

define functions of y which, for any small $|m|$, are holomorphic in the neighborhood of $y = 0$. Supposing now that m and y vary, let us follow the variation of the two functions (8). In order to do that, we shall be obliged, as $|y|$ or $|m|$ grows, to alter the figure of the curves Γ_1 and Γ_2; we have namely to submit these curves to a continuous deformation which makes it possible to avoid encountering on them any critical point of the branch z integrated (in the right-hand members of (8)). As long as such a continuous deformation is possible, the functions $\psi_1(y)$ and $\psi_2(y)$ cannot cease to be holomorphic. But in only one case will the deformation become impossible: namely when the curve Γ_1 or Γ_2 has to pass *between*

two critical points of z which come to coincide; in this last case, Γ_1 or Γ_2 *must* cross the multiple singular point, which is bound to be one of the three transcendent singularities of the differential equation, namely

$$x = \alpha = e^{2i\pi/3}; \qquad x = \beta = e^{-(2i\pi/3)}; \qquad x = \gamma = 1 \qquad \text{(see 3).}$$

In order to make the full discussion which follows geometrically simple, we shall assume, henceforth, that m_1 in equations (4) and (5), is *a negative real number of small absolute value* (see 2). This value of m is now supposed to be fixed while we move y on all straight lines starting from $y = 0$. On this set of lines of the y-plane, we get *one* special branch of $\psi_1(y)$ and *one* special branch of $\psi_2(y)$ on which we shall first and *provisionally* fix our attention.

Under these conditions, I prove that, following the two ψ's on the said straight lines, I find only:

Two singularities of $\psi_1(y)$, namely one, y_α, for which Γ_1 crosses $x = \alpha$, and one, y_γ, for which Γ_1 crosses $x = \gamma$;

Two singularities of $\psi_2(y)$, namely one, y_β, for which Γ_2 crosses $x = \beta$, and the other, y_γ (the same as y_γ as above), for which Γ_2 crosses $x = \gamma$.

To these singularities correspond:

Two singularities of $\psi_1^{(-1)}(y)$, the inverse function of ψ_1, namely $y_{\alpha 1}$ and $y_{\gamma 1}$ such that $y_{\alpha 1} = \psi_1(y_\alpha)$, $y_{\gamma 1} = \psi_1(y_\gamma)$;

Two singularities of $\psi_2^{(-1)}(y)$, the inverse function of ψ_2, namely $y_{\beta 2} = y_2(y_\beta)$ and $y_{\gamma 2} = \psi_2(y_\gamma)$.

As for the position of the points y_α, etc., we easily see, that, if $|m|$ is small (as assumed), y_α and $y_{\alpha 1}$ are close to the value $3/2e^{2i\pi/3}$ which they have for $m = 0$; y_β and $y_{\beta 2}$ are close to $3/2e^{-(2i\pi/3)}$; y_γ, $y_{\gamma 1}$ and y_γ^2 are close to $3/2$; furthermore, if m is real and negative, y_γ is real and positive while the couples y_α and y_β, $y_{\alpha 1}$ and $y_{\beta 2}$, $y_{\gamma 1}$ and $y_{\gamma 2}$ are conjugate imaginaries, y_α, $y_{\alpha 1}$ and $y_{\gamma 2}$ being above the real x-axis.

Besides, whatever be m, we have

$$y_{\gamma 2} - y_{\gamma 1} = 2i\pi\eta_1,$$

η_1 being the number defined in section 3 above (see expansion (6)).

7. **Definition of fields $F_1, \ldots F_2$ and of a set of fundamental substitutions (S).** Let us draw and name the following lines in the y-plane:

$0y_\gamma\infty$, real positive half-axis from $y = 0$ to $y = \infty$;

$0_1y_{\gamma 1}\infty$ and $0_2y_{\gamma 2}\infty$, transformed of $0y_\gamma\infty$ by the substitutions $[y, \psi_1(y)]$, $[y, \psi_2(y)]$, *where we take for ψ_1 and ψ_2 the branches of the ψ-functions previously defined in* 6.

$0y_\alpha$ and $0y_\beta$ straight lines;

$0_1y_{\alpha 1}$ and 0_1y_β transformed of $0y_\alpha$ by the substitution $[y, \psi_1]$ and $[y, \psi_2]$ defined as above;

O_2y_α and $O_2y_{\beta 2}$, transformed of Oy_β by the same substitutions.

The lines so defined are shown in Fig. 2. If m is real, negative and small (as assumed), all these lines are approximately straight (when not exactly); the line $y_\alpha O_2 y_{\gamma 2}\infty$ is above the line $y_\alpha O y_\gamma \infty$ and does not cut that line, nor cut itself; furthermore $y_\alpha O_2 y_{\gamma 2}\infty$ and $y_\beta O_1 y_{\gamma 1}\infty$ are symmetrical with respect to the real axis; so are also $O_1 y_{\alpha 1}$ and $O_2 y_{\beta 2}$.

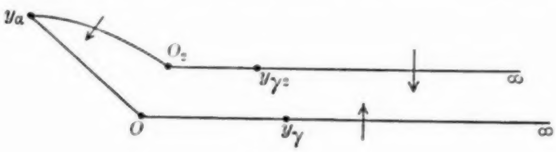

FIG. 2.

I shall now call $F_1, \ldots F_2$ the fields in the y-plane thus defined:

F_1 is the outside of the closed curve $\infty\, y_\gamma O y_\alpha O_2 y_{\gamma 2}\,\infty$;

F_2 is the outside of the closed curve $\infty\, y_\gamma O y_\beta O_1 y_{\gamma 1}\,\infty$;

F_{-1} is the whole y-plane *cut* by the line $y_{\alpha 1} O_1 y_{\gamma 1}\,\infty$;

F_{-2} is the whole y-plane *cut* by the line $y_{\beta 2} O_2 y_{\gamma 2}\,\infty$.

In the respective fields $F_1, \ldots F_{-2}$, let us consider the functions ψ_1, ψ_2, and the inverse functions $\psi_1^{(-1)}$, $\psi_2^{(-1)}$, and, more precisely those determinations of the four functions which are such that $\psi_1(y_\alpha) = y_{\alpha 1}$, $\psi_2(y_\beta) = y_{\beta 2}$; $\psi_1^{(-1)}(y_{\alpha 1}) = y_\alpha$; $\psi_2^{(-1)}(y_{\beta 2}) = y_\beta$. Of each of the four functions I have one single uniform and holomorphic branch or *element* in the corresponding field. Let us call $\hat{\psi}_1, \ldots \hat{\psi}^{(-2)}$ the uniform elements of functions so defined (it will be noticed that, according to the definition of the fields F, $\hat{\psi}_1^{(-1)}$ and $\hat{\psi}_2^{(-2)}$ are defined for all values of y, while $\hat{\psi}_1$ and $\hat{\psi}_2$ are not).

The elements of functions $\hat{\psi}_1 \ldots \hat{\psi}_2^{(-2)}$ are analytically defined by equations (8) and, being uniform, may be easily represented by convergent expansions in their respective fields. From the point of view of Painlevé, they are "known functions."

We have to remark that fields the F_1 and F_{-1} (or F_2 and F_2^{-1}) do not correspond exactly to each other: in other words, when y describes the boundary of F_1, the corresponding point $\hat{\psi}_1(y)$ does not describe the boundary of F_{-1} from end to end. If, however, we superpose two y-planes one bearing F_1 and F_2 and the other F_{-2} and F_{-1} and consider in the two sheets two regions, composed, one of F_1 and F_{-2}, the other of F_{-1} and F_2, we have an exact correspondence between these two regions. In other words, let us call $\mathcal{F}_1$, on the two-sheets, the field bounded by the line $\infty\, y_\gamma O y_\alpha O_2$ (*first sheet*) $O_2 y_{\beta 2} O_2 y_{\gamma 2}\,\infty$ (*second sheet*); let us call $\mathcal{F}_2$ the field bounded by the line $\infty\, y_\gamma O y_\beta O_1$ (*first sheet*), $O_1 y_{\alpha 1} O_1 y_{\gamma 1}\,\infty$ (*second sheet*). When the point y, starting from y_α (first sheet) covers the whole area $\mathcal{F}_1$, the corresponding

point $\psi_1(y)$ starting from y_{a1} (second sheet) covers the whole area $\mathcal{F}_2$ (here, of course, $\psi_1(y)$ coincides with $\hat{\psi}_1(y)$ in one part of $\mathcal{F}_1$ only; in the remaining part, it coincides with $\hat{\psi}_2^{(-1)}(y)$, as may be deduced from the facts stated in section 9 below).

8. **Solution of the group-problem.** The "elements of functions" $\hat{\psi}_1(y)$, $\hat{\psi}_2(y)$, ... $\hat{\psi}_2^{(-1)}(y)$ being defined as stated in 7 (in their respective fields), let us call (S_1), ... (S_2^{-1}) the substitutions $(S_1) = [y, \hat{\psi}_1(y)]$, $(S_2) = [y, \hat{\psi}_2(y)]$, ... $(S_2^{-1}) = [y, \hat{\psi}_2^{(-1)}(y)]$. To these substitutions and inverse substitutions we add a third

$$(S_3) = [y, y + 2i\pi\eta_1], \qquad (S_3^{-1}) = [y, y - 2i\pi\eta_1]$$

which is uniformly defined for every y (η_1 is the number defined in 4). This being done, I come to the following result which I will submit here without proof: *All substitutions* (S) *of group* G *(see 5) are products of the three substitutions* (S_1), (S_2), (S_3) *and the inverse substitutions.*

In other words, *with the three fundamental substitutions* (S_1), (S_2), (S_3) *(and inverse substitutions) defined by uniform elements of* ψ*-functions we can build up the total group* G.

Reciprocally, of course, any product of (S_1), ... (S_3^{-1}) is a substitution of G. But we have to remember the fact that for some values of y it will happen that one of the fundamental substitutions will cease to exist. Namely (S_1) is *not* defined for y inside of the curve $\infty 0 y_\gamma y_a y_{\gamma 3} \infty$ (see Figure 2), and (S_2) is *not* defined for y inside of the curve $\infty 0 y_\beta y_\gamma \infty$.

The multiplication of substitution (S_1), ... (S_3^{-1}), it will be noticed, is *not commutative.*

9. **Automorphic properties of the ψ-functions.** The ψ-functions are multiform functions which have the property that the knowledge of two branches or "elements" of these functions is sufficient to get the whole functions (in their whole field of existence) by means of multiplication of substitutions. In other words, any branch of a ψ-function is made up of a few fundamental branches combined together. This character may be described as an automorphic property of the function.

In order to verify that the ψ-functions actually have this property, we have to show this: The initial elements of the ψ's have been defined in certain fields F_1, ... F_2^{-1} of the y-plane; we have to show that whenever we cross a boundary of F_1, F_2, ... or F_2^{-1}, the corresponding substitution (S_1), (S_2), ... (S_2^{-1}), which then ceases to be defined as one of the fundamental substitutions, *will be transformed into some product of the fundamental substitutions.*

Let us, for instance, state here what becomes of (S_1) when we cross the boundaries of F_1. I prove that:

if we cross the line $y_\gamma\ \infty$ (coming from the inside of F_1, that is from below on the diagram), $(S)_1$ is changed for $(S_2 . S_3^{-1})$;

if we cross the line $y_\gamma 0 y_\alpha$, (S_1) is changed into $(S_2 . S_1)$;

if we cross the line $y_\alpha 0_2$ (coming from F_1), (S_1) is changed into $(S_2^{-1} . S_1)$;

if we cross the line $0_2 y_{\alpha 2}\ \infty$, (S_1) is changed into (S_2^{-1}).

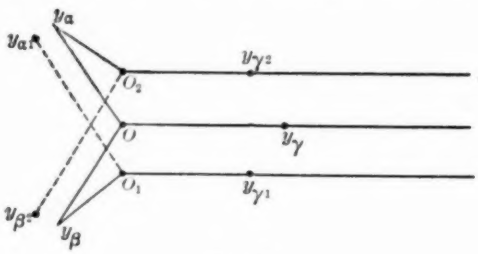

FIG. 3.

All the above combinations of substitutions are *defined* (*exist*) for the value of y to which they are applied. For instance, let us consider $(S_2 . S_1)$ in the points y which are met when we cross $y_\gamma 0 y_\alpha$, that is just above the line $y_\gamma 0 y_\alpha$ on the diagram: (S_2) operating on such points carries us *above* the line $y_\alpha 0_2 y_{\gamma 2}$, where (S_1) is defined.

It will be noticed, that, considered in their whole field of existence, the four functions $\psi_1, \ldots \psi_2^{(-1)}$ consist of only one function and the inverse function, as follows from the fact that ψ_1 is changed into $\psi_2^{(-1)}$ when y describes a certain path.

10. **Further questions.** The definition and the properties of the ψ-functions suggest the investigation of some other connected functions defined by integrals of the form $\int dx/z$ (z an integral of the above differential equation) which I shall try to consider in another paper. On the other hand the results which I have stated in the case of equation (4) have to be extended to the most general equation (1) of type A (see 1) and first, of all, to equation

$$zz' = 3mz + ax^3 + bx^2 + cx + d \tag{9}$$

for any values of the coefficients m, a, b, c, d.

In this respect, I may remark, that, while it is easy to make this extension, and thus to find *one solution* of the group-problem for the general equation (9), it is more difficult to find the *simplest* solution, which will be quite different for different values of $m, a, \ldots d$. In other words, we always have a group G which we can build up with a limited number of fundamental substitutions, but the choice of the substitutions which it will be most convenient to take as the fundamental ones (and for which

the *fields* called F_1, F_2, ..., etc. will be simplest), will be different for different kinds of equations (9).

One interesting case is a case of which we have an instance by making $m = 1$ in equation (4). For the equation

$$zz' = 3z + 2(x^3 - 1) \tag{10}$$

the number η_1 (equal to $6m(m^3 - 1)$ is 0 and substitution (S_3) disappears. In this case, it is actually possible to build up group G with only the two substitutions (S_1), (S_2), and the inverse substitution, these substitutions being defined in fields similar to those considered in section 7. Equation (10) has, among its solutions, one particular solution which is the polynomial $z = x^2 + x + 1$.

It will be noticed, that, while $\eta_1 = 0$, the number called η_2 (see 4) is not 0 for equation (10). An equation (9) for which both η_1 and η_2 are 0 at the same time would be an equation having, among its solutions, two polynomials, for instance

$$zz' = 3z + 2x(x - 1)(x - 2),$$

which is satisfied by $z = x(x - 1)$ and $z = -(x - 1)(x - 2)$. It is easily proved that such a differential equation can be solved in terms of elliptic functions and does not, therefore, define any new functions.

HERMITIAN METRICS.

By J. L. Coolidge.

Introduction. The smoke of battle that used to surround the various Non-Euclidean geometries has passed away and we are at present able to appreciate the underlying significance of much that was formerly mysterious and profited, to a certain extent, by the charm that attaches to mystery. To the abstract logician the Non-Euclidean problem comes under the general head of the independent axiom problem. To the working mathematician, assuming for the sake of argument that such a person exists, the matter presents itself in about the following terms:

"Here is a collection of objects which we call points. Each is possessed of a set of coördinates, we do not care whether they were honestly acquired or not. Here is a function which we call the distance of two points. What can be said about the system?"

In the classical Non-Euclidean systems the function is obtained by polarizing a given quadratic form, and these geometries will doubtless always remain the most important; but endless other methods of procedure are conceivable, and all are not entirely barren of interesting results. Perhaps the most successful venture of this sort consists in using, as the basis of measurement, a so-called "Hermitian form." This is an expression of the type

$$\Sigma a_{ij} x_i \bar{x}_j \qquad a_{ji} = \bar{a}_{ij} \qquad (i, j = 0, 1, \cdots n),$$

where any letter with a dash over the top is supposed to be the conjugate imaginary of the same letter without the dash. It is usual also to consider only the case where the discriminant is not zero, and to define as the distance of two points (x) (y) the number d defined by the equation

$$\cos\frac{d}{k} = \frac{\sqrt{\Sigma a_{ij} x_i \bar{y}_j}\,\sqrt{\Sigma a_{ij} y_i \bar{x}_j}}{\sqrt{\Sigma a_{ij} x_i \bar{x}_j}\,\sqrt{\Sigma a_{ij} y_i \bar{y}_j}}.$$

It is not difficult to show that by a properly chosen collineation our Hermitian form can be reduced to the canonical type

$$\Sigma \pm x_i \bar{x}_i.$$

Of all possible geometries based upon such forms, the richest and most symmetrical is the so called "elliptic" type where all of the terms above

are positive. If we use the common abbreviation

$$(ab) = \Sigma a_i b_i$$

the fundamental distance formula is

$$\cos d = \frac{\sqrt{(x\bar{y})}\sqrt{(\bar{x}y)}}{\sqrt{(x\bar{x})}\sqrt{(y\bar{y})}}.$$

The next most interesting type is a limiting case of this. Let us replace d by d/k, and x_0 by kx_0. Then

$$k \sin\frac{d}{k} = \frac{\sqrt{\sum_{i=0}^{i=n}(x_i y_0 - x_0 y_i)(\bar{x}_i\bar{y}_0 - \bar{x}_0\bar{y}_i) + \frac{1}{2k^2}\sum_{i,j=1}^{i,j=n}(x_i y_j - x_j y_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i)}}{\sqrt{x_0\bar{x}_0 + \frac{1}{k^2}\left(\sum_{i=1}^{i=n} x_i\bar{x}_i\right)}\sqrt{y_0\bar{y}_0 + \frac{1}{k^2}\left(\sum_{i=1}^{i=n} y_i\bar{y}_i\right)}}$$

and the limit of this as k becomes infinite is

$$d = \frac{\sqrt{\sum_{i=1}^{i=n}(x_i y_0 - x_0 y_i)(\bar{x}_i\bar{y}_0 - \bar{x}_0\bar{y}_i)}}{\sqrt{x_0\bar{x}_0}\sqrt{y_0\bar{y}_0}},$$

which looks more familiar in the non-homogeneous form*

$$d = \sqrt{(x'-x)(\bar{x}'-\bar{x}) + (y'-y)(\bar{y}'-\bar{y}) + (z'-z)(\bar{z}'-\bar{z}) + \cdots}.$$

1. The metric of a single line. Let us fix a point on a line by two homgeneous coördinates x_0, x_1 not both simultaneously zero.

The distance of two points shall be given by the equation

$$\cos d = \frac{\sqrt{x_0\bar{y}_0 + x_1\bar{y}_1}\sqrt{\bar{x}_0 y_0 + \bar{x}_1 y_1}}{\sqrt{x_0\bar{x}_0 + x_1\bar{x}_1}\sqrt{y_0\bar{y}_0 + y_1\bar{y}_1}}. \qquad (1e)$$

To find a geometrical meaning for this let us represent the totality of points of our line by the real points of a Riemann sphere†

$$X = \frac{x_0\bar{x}_1 + x_1\bar{x}_0}{x_0\bar{x}_1 + x_1\bar{x}_1}, \qquad Y = i\frac{x_0\bar{x}_1 - x_1\bar{x}_0}{x_0\bar{x}_0 + x_1\bar{x}_1}, \qquad Z = \frac{x_0\bar{x}_0 - x_1\bar{x}_1}{x_0\bar{x}_0 + x_1\bar{x}_1},$$

$$X^2 + Y^2 + Z^2 = 1.$$

* The present author is familiar with only two articles on the Hermitian metric. The first of these was a slight contribution by Fubini "Sulle metriche definite da una forma Hermitiana," Atti della R. Istituto Veneto, vol. LXIII, 1903–04. Far different is the fundamental article by Study "Kurzeste Wege im komplexen Gebiete," Math. Annalen, vol. 60, 1905. The parabolic case is here treated in a stepmotherly fashion, but the elliptic one receives long attention. Yet Study, curiously enough, does not touch upon the most immediate and elementary questions of the Hermitian metric, so that there is very little overlapping between his work and the present paper.

† We shall habitually use large letters to signify real values, and small ones for complex ones.

If the complex point y_0, y_1 corresponds to the real point (X', Y', Z') and Θ is the angle subtended at the center of the sphere by the points $(X, Y, Z)(X', Y', Z')$

$$\cos\frac{\Theta}{2} = \sqrt{\frac{XX' + YY' + ZZ' + 1}{2}}$$

$$= \frac{\sqrt{x_0\bar{y}_0 + x_1\bar{y}_1}\sqrt{\bar{x}_0 y_0 + \bar{x}_1 y_1}}{\sqrt{x_0\bar{x}_0 + x_1\bar{x}_1}\sqrt{y_0\bar{y}_0 + y_1\bar{y}_1}}$$

$$= \cos d.$$

THEOREM 1*e*. *The elliptic Hermitian distance of two points is one half the arc of the great circle connecting the corresponding points of the Riemann sphere.**

Passing over to the parabolic case we take the distance formula

$$d = \frac{\sqrt{(x_0 y_1 - x_1 y_0)}\sqrt{(\bar{x}_0\bar{y}_1 - \bar{x}_1\bar{y}_0)}}{\sqrt{x_0\bar{x}_0}\sqrt{y_0\bar{y}_0}} \tag{1p}$$

$$= \sqrt{(x' - x)(\bar{x}' - \bar{x})}.$$

If we write

$$x = X + iY, \qquad x' = X' + iY',$$

$$d = \sqrt{(X' - X)^2 + (Y' - Y)^2}.$$

THEOREM 1*p*. *The parabolic Hermitian distance of two points is the absolute value of their Euclidean distance, and also the distance of the two points which represent them in the Gauss plane.*

Let A, B, C be three points. When shall we have the equation

$$AC + CB = AB?$$

We see at once that, in the elliptic case, the real points which represent the given complex ones must lie on a great circle, and in the parabolic case the representing points must be collinear. Also AB must be the largest of the three distances.

Definition. A system of collinear points of such a nature that the cross ratio of any four is real, shall be said to belong to a *chain.*†

A chain connecting the points (X') and (X'') can always be represented in the form

$$x_i = \Xi_1 x_i' + \Xi_2\rho x_i'', \qquad \bar{x}_i = \Xi_1\bar{x}_i' + \Xi_2\bar{\rho}\bar{x}_i'', \qquad i = 0, 1, \cdots n. \tag{2}$$

* Study, loc. cit., p. 333, writes $d/2$ where we write d, so as to identify distance on the complex line with arc on the Riemann sphere.

† The literature of chains is large. The original idea is due to Von Staudt. See his "Beiträge zur Geometrie der Lage," Part 2, Nuremberg, 1853 ±.

If, in the elliptic case, we have the additional relation

$$\bar{\rho}(x'\bar{x}'') = \rho(\bar{x}'x''),$$

the chain is said to be a *normal* one. The equations for a normal chain are simplified if we imagine the complex multiplier ρ swallowed into the homogeneous coördinates (x''). We have, then

$$\begin{gathered} x_i = \Xi_1 x_i' + \Xi_2 x_i'', \qquad \bar{x}_i' = \Xi_1 \bar{x}_i' + \Xi_2 \bar{x}_i'', \\ (x'\bar{x}'') = (\bar{x}'x''). \end{gathered} \tag{3}$$

In the one-dimensional elliptic case if we now put

$$\Xi_1' = (\bar{x}_0'x_0'' + \bar{x}_1'x_1'')\Xi_1 + (x_0''\bar{x}_0'' + x_1''\bar{x}_1'')\Xi_2,$$

$$\Xi_2' = -(x_0'\bar{x}_0' + x_1'\bar{x}_1')\Xi_1 - (x_0'\bar{x}_0'' + x_1'\bar{x}_1'')\Xi_2,$$

we carry our normal chain into itself while we replace x_0, x_1 by $\bar{x}_1$, $-\bar{x}_0$ and X, Y, Z by $-X$, $-Y$, $-Z$, so that each great circle is invariant under the transformation. Hence our normal chain corresponds to a great circle. For a normal chain in the parabolic case we write

$$x_0'\bar{x}_0'' = \bar{x}_0'x_0''.$$

We can find a point on the chain for which the first coördinate is zero, i.e., the chain contains the infinite point on the line, and is represented in the Gauss plane by a straight line.

THEOREM 2. *The necessary and sufficient condition that three collinear points should be connected by a relation*

$$AC + CB = AB$$

is that they should belong to one normal chain, and that the last of the three distances determined by them should be the greatest.

2. **Plane Trigonometry.** We next suppose that we are dealing with points in one plane. For the distance of two points we have, in the elliptic case

$$\cos d = \frac{\sqrt{(x\bar{y})}\sqrt{(y\bar{x})}}{\sqrt{(x\bar{x})}\sqrt{(y\bar{y})}}, \tag{4e}$$

whereas in the parabolic case we have

$$d = \sqrt{(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y})}. \tag{4p}$$

With regard to the first of these formulæ we notice that

$$1 - \cos^2 d = \frac{\frac{1}{2}\Sigma(x_i y_j - x_j y_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i)}{(x\bar{x})(y\bar{y})} > 0.$$

If the equation of a line be written in the compact form

$$(ux) = 0$$

we shall define the elliptic angle of two lines (u) and (v) by the equation

$$\cos\theta = \frac{\sqrt{(u\bar{v})}\sqrt{(\bar{u}v)}}{\sqrt{(u\bar{u})}\sqrt{(v\bar{v})}}. \tag{5e}$$

In the parabolic case we shall adopt the simpler form

$$\cos\theta = \frac{\sqrt{u_1\bar{v}_1 + u_2\bar{v}_2}\sqrt{\bar{u}_1v_1 + \bar{u}_2v_2}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\sqrt{v_1\bar{v}_1 + v_2\bar{v}_2}}. \tag{5p}$$

We shall define the distance from a point to a line, as the distance to the foot of the perpendicular on the line. We thus get, for the point (x) and the line (u) the twin formulas

$$\sin d = \frac{\sqrt{(ux)}\sqrt{(\bar{u}\bar{x})}}{\sqrt{(u\bar{u})}\sqrt{(x\bar{x})}}, \tag{6e}$$

$$d = \frac{\sqrt{(ux)}\sqrt{(\bar{u}\bar{x})}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\sqrt{x_0\bar{x}_0}}. \tag{6p}$$

For real elements, the elliptic form of Hermitian measurement is identical with that for a projective plane in elliptic space of total curvature unity, while the parabolic Hermitian measurement is identical with the usual Euclidean form.

A collineation which leaves distances invariant shall be said to be "congruent." An elliptic congruent collineation may be written*

$$\rho x_i' = a_i x_0 + \frac{[(\bar{a}b)a_i - (a\bar{a})b_i]}{\sqrt{(a\bar{a})(b\bar{b}) - (a\bar{b})(\bar{a}b)}}x_1 + \frac{\sqrt{(a\bar{a})}(a_jb_k - a_kb_j)}{\sqrt{(a\bar{a})(b\bar{b}) - (a\bar{b})(\bar{a}b)}}x_2 \tag{7e}$$

The parabolic collineation takes the somewhat simpler form

$$\begin{aligned} \rho x_0' &= a_0x_0 \\ \rho x_1' &= a_1x_0 + \cos A e^{i\Theta_1}x_1 + \sin A e^{i(\Theta_1+\Phi)}x_2, \\ \rho x_2' &= a_2x_0 \pm \sin A e^{i\Theta_2}x_1 \mp \cos A e^{i(\Theta_2+\Phi)}x_2. \end{aligned} \tag{7h}$$

We see that the point $(1, 0, 0)$ is carried into the arbitrary point (a). Hence, the transformation is transitive. On the other hand, if $(1, 0, 0)$

* The first writer to give implicitly the form for a congruent collineation of the elliptic type was Loewy. See his "Über bilineare Formen mit konjugirt imaginären Variabeln," Nova Acta Leopoldina, vol. 71, 1898. The explicit form, was first given by the author in his article "The Geometry of Hermitian Forms," Transactions American Mathematical Society, vol. 21, 1920.

is fixed, we have, in both cases, the same type of rotation, which will carry any line through the fixed point into any other such line. Lastly, if a point and a line through it be fixed (which fixes the perpendicular through the point), there is still possible a group of congruent collineations depending on two real parameters namely

$$\rho x_i' = e^{\theta_i} x_i. \tag{8}$$

Definition. A system of concurrent and coplanar lines of such a sort that the cross ratio of any four is real is said to belong to a "line chain."

The general analytic expression for a chain will be:

$$u_i = H_1 u_i' + H_2 \rho u_i''. \tag{9}$$

This chain, in the elliptic case, will be defined as *normal* if

$$\bar{\rho}(u'\bar{u}'') = \rho(\bar{u}'u''). \tag{10e}$$

Since distance and angle formulas are the same in the elliptic case, this shows that, if a, b, c be three lines of the chain, there will always be a relation of the form:

$$\angle\, ac + \angle\, cb = \angle\, ab.$$

A similar result will hold in the parabolic case if the terms with subscript zero be suppressed. A line chain will always cut a transversal in a chain of points; when will a normal line chain cut a normal point chain? Let the transversal be the line $x_2 = 0$, while the vertex of the line pencil is $(1, 0, c)$. A typical line of the chain may be written

$$H_1(acx_0 - cx_1 - ax_2) + H_2\rho[bcx_0 - cx_1 - bx_2] = 0.$$

Since the line chain is by hypothesis, normal, we have in the elliptic case

$$\bar{\rho}[c\bar{c}(a\bar{b} + 1) + a\bar{b}] = \rho[c\bar{c}(\bar{a}b + 1) + \bar{a}b].$$

On the other hand, the point chain is expressed

$$x_0 = H_1 c + H_2 \rho c, \qquad x_1 = H_1 ac + H_2 \rho bc;$$

and this will be normal if

$$\bar{\rho}c\bar{c}[a\bar{b} + 1] = \rho c\bar{c}[\bar{a}b + 1].$$

The two conditions are compatible when, and only when

$$\rho = \bar{\rho}, \qquad a\bar{b} = \bar{a}b.$$

Exactly similar reasoning is applicable in the parabolic case, and we have

THEOREM 3. *The necessary and sufficient condition that a normal line chain should cut a transversal in a normal chain, is that the perpendicular*

from the vertex of the line chain upon the transversal should be included in the line chain. The necessary and sufficient condition that a normal point chain should determine a normal chain about a given point, is that the foot of the perpendicular from the given point upon the line of the chain should belong to the line chain.

The two equations just written express the necessary and sufficient condition that there should exist a congruent transformation of the type (8) which carries the three sets of coördinate values $(1, 0, c)$, $(c, ac, 0)$, $(\rho c, \rho bc, 0)$, into three real sets:

THEOREM 4. *A normal line chain will intersect a transversal in a normal point chain, and a normal point chain will determine a normal line chain about a given point, when, and only when, there exists a congruent collineation which transforms the two simultaneously into real point and line chains.*

We shall presently see that the perpendicular from a point upon a line acts as a universal solvent in many trigonometric problems. Let us first look for the formulas for the right triangle. By a proper change of axes we may suppose that if we have a triangle right-angled at C we may take for the coördinates of the vertices $(1, 0, 0)$, $(1, a, 0)$, $(1, 0, c)$; while the sides have the coördinates $(-ac, c, a)$, $(0, 0, 1)$, $(0, 1, 0)$. In the elliptic case

$$\cos AB = \frac{1}{\sqrt{1 + a\bar{a}}\sqrt{1 + c\bar{c}}} = \cos BC \cos CA,$$

$$\sin BC = \frac{\sqrt{c\bar{c}}}{\sqrt{1 + c\bar{c}}} = \frac{\sqrt{a\bar{a}c\bar{c} + a\bar{a} + c\bar{c}}}{\sqrt{1 + a\bar{a}}\sqrt{1 + c\bar{c}}} \cdot \frac{\sqrt{c\bar{c} + a\bar{a}c\bar{c}}}{\sqrt{a\bar{a}c\bar{c} + a\bar{a} + c\bar{c}}}$$

$$= \sin AB \sin A.$$

Exactly similar formulas hold in the parabolic case:

THEOREM 5. *The trigonometric formulas for a right triangle are the same in the Hermitian metrics as in the corresponding Non-Euclidean or Euclidean metrics.*

We see that the hypotenuse of a right triangle is always greater than one leg, and a side of a triangle is greater than its projection on another side:

THEOREM 6. *If A, B, C, be any three points the distance AB is never greater than the sum of the distances AC and CB and is only equal to that sum when (a) the points are collinear, (b) they belong to a normal chain, and (c) AB is the greatest of the three distances.**

The normal chain is thus a geodesic in our plane, i.e., the shortest path between two given points.

* For an algebraic proof see Study, loc. cit., p. 330ff.

If an arbitrary triangle be given, we may take as the coördinates of its vertices the values $(1, 0, c)$, $(1, a, 0)$, $(1, b, 0)$. These points may be carried simultaneously by a congruent collineation into real points when, and only when

$$a\bar{b} = \bar{a}b.$$

When this relation holds, it is evident that the usual elliptic or Euclidean trigonometric relations must hold between the sides and angles of the triangle. Conversely, if such relations hold, the foot of an altitude must lie on a normal chain with the two vertices collinear therewith, and we can assume that the vertices have these coördinates and that this equation is true. We may, however, put the matter still more concretely. Two sides of our triangle have the equations

$$-acx_0 + cx_1 + ax_2 = 0, \qquad -bcx_0 + cx_1 + bx_2 = 0.$$

The perpendiculars on these from the opposite vertices have, in the elliptic case, the equations

$$a\bar{b}x_0 - \bar{b}x_1 + \bar{c}(1 + a\bar{b})x_2 = 0, \qquad \bar{a}bx_0 - \bar{a}x_1 + \bar{c}(1 + \bar{a}b)x_2 = 0.$$

These will be concurrent upon the third altitude when, and only when

$$a\bar{b} = \bar{a}b.$$

THEOREM 7. *The necessary and sufficient condition that the Hermitian trigonometry of a triangle be that of the corresponding Non-Euclidean or Euclidean system of metrics, is that the altitudes should be concurrent. In this case, and in this case only, the foot of one altitude, and, hence of each altitude, is on the normal chain determined by the vertices collinear with it. In this case, and in this case only, it is possible to carry the triangle by a congruent collineation into a real triangle.*

3. **Hyperconics.** Let us find the locus of a point at a given distance from a given point (y). In the elliptic case we have, clearly

$$(x\bar{y})(\bar{x}y) - \cos^2 d(x\bar{x})(y\bar{y}) = 0,$$

and in the parabolic one

$$(x_1y_0 - x_0y_1)(\bar{x}_1\bar{y}_0 - \bar{x}_0\bar{y}_1) + (x_2y_0 - x_0y_2)(\bar{x}_2\bar{y}_0 - \bar{x}_0\bar{y}_2) - d^2x_0\bar{x}_0y_0\bar{y}_0 = 0,$$

which is written non-homogeneously

$$(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y}) = d^2.$$

In the former case, if $d = 0$ we have

$$\Pi(x_iy_j - x_jy_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i) = 0$$

so that the point (y) alone lies on the locus, whereas if $d = \pi/2 \pmod{\pi}$ we have merely the straight line

$$(\bar{y}x) = 0.$$

We shall call the general locus a "hypercircle."

THEOREM 8. *The totality of points at a given distance from a given point will depend upon three real parameters, except in the case where the distance is zero and there is but one point in the totality, or when, in the elliptic metric it is equal to* $\pi/2 \pmod{\pi}$ *and the totality consists in the points of a line. In the general case the equation of the locus is expressed by equating a Hermitian form to zero.*

If we place the point (1, 0, 0) at the center of the hypercircle, the equation takes one of the canonical forms

$$-\tan^2 d x_0\bar{x}_0 + x_1\bar{x}_1 + x_2\bar{x}_2 = 0, \tag{11e}$$

$$x\bar{x} + y\bar{y} = r^2. \tag{11p}$$

The polar of a given point with regard to the hypercircle will have the equation

$$-\tan^2 d\bar{y}_0 x_0 + \bar{y}_1 x_1 + \bar{y}_2 x_2 = 0, \tag{12e}$$

$$\bar{x}'x + \bar{y}'y = r^2. \tag{12p}$$

THEOREM 9. *The polar of a point with regard to a hypercircle is perpendicular to the line connecting the given point with the center. In the elliptic case the product of the tangents of the distances of the center of a hypercircle from a point and from its polar is equal to the square of the tangent of the radius, in the parabolic case the product of the distances is the radius squared.*

THEOREM 10. *If the pole of a line be at more than a radius distance from the center, the line meets the locus in a chain of points at the same distance from the foot of the perpendicular; if a point be on a hypercircle, its polar meets the hypercircle in that point and nowhere else, if the point be at less than a radius distance from the center, its polar contains no point of the hypercircle.*

A line meeting a hypercircle in a single point may be defined as a "tangent" thereto. It is not, however, the limiting position towards which a secant necessarily approaches as two points of intersection tend to coalesce. If we remember that the formulas for distance and angle are entirely analogous in the elliptic case, and that the tangential equation of a hypercircle is obtained from its point equation by exactly the same process as is used for a circle, we reach:

THEOREM 11e. *In the elliptic Hermitian metric, the lines which meet a*

fixed line at a fixed angle, which is not a right angle, will be tangent to a hypercircle whose center is the pole of the fixed line.

THEOREM 11p. *In the parabolic Hermitian metric the lines which meet a fixed line at a fixed angle other than a right angle will be parallel to the lines of a chain.*

Suppose that we have an equation obtained by setting a Hermitian form of non-vanishing discriminant equal to zero

$$\sum_{i,j=0}^{i,j=2} a_{ij}x_i\bar{x}_j = 0, \qquad |a_{ij}| \neq 0, \qquad a_{ji} = \bar{a}_{ij}. \tag{13}$$

If there be a single point whose coördinates satisfy this equation, there will be a system depending on three real parameters. We shall call the corresponding locus a "hyperconic." We define as the polar of a point (y) the line

$$\Sigma a_{ij}\cdot x_i\bar{y}_j = 0.$$

If (y) be a point of the hyperconic, it lies on its polar, which will contain no other point of the hyperconic, and which we shall call a "tangent" thereto. The tangential equation of the hyperconic is

$$\Sigma A_{ij}\cdot u_i\bar{u}_j = 0.$$

Let us seek a canonical form for the equation. We write the characteristic equation for the elliptic case

$$\begin{vmatrix} a_{00} - \rho & a_{01} & a_{02} \\ \bar{a}_{01} & a_{11} - \rho & a_{12} \\ \bar{a}_{02} & \bar{a}_{12} & a_{22} - \rho \end{vmatrix} = 0.$$

This equation has surely one root, so that there is one point which has the same polar with regard to the given Hermitian form and with regard to the form which is the basis of our elliptic measurement. Taking this as $(1, 0, 0)$ and giving to its common polar with regard to these two forms these same coördinates, the equation of the hyperconic becomes

$$a_{00}x_0\bar{x}_0 + a_{11}x_1\bar{x}_1 + a_{12}x_1\bar{x}_2 + \bar{a}_{12}\bar{x}_1x_2 + a_{22}x_2\bar{x}_2 = 0.$$

Consider the reduced characteristic equation

$$\begin{vmatrix} a_{11} - \rho & a_{12} \\ \bar{a}_{12} & a_{22} - \rho \end{vmatrix} = 0.$$

This will have equal roots if

$$(a_{11} - a_{22})^2 + 4\, a_{12}\bar{a}_{12} = 0,$$

two equations which can be satisfied only if

$$a_{11} = a_{22}, \qquad a_{12} = \bar{a}_{12} = 0,$$

and we have the hypercircle

$$a_{00}x_0\bar{x}_0 + a_{11}(x_1\bar{x}_1 + x_1\bar{x}_2) = 0.$$

In the general case this equation will have at least one root different from a_{00}, giving a second point with the same polar with regard to the two Hermitian forms. Taking this pole and its polar as (0, 1, 0) we find the canonical form for our hyperconic

$$\Sigma A_i x_i \bar{x}_i = 0. \tag{14}$$

Since this equation must not be an absurdity, we may assume that two of the A_i's are positive, and the third negative. Each vertex of the coördinate triangle is a "center" in the sense that a line through it which meets the hyperconic does so in pairs of points equidistant from the center, and on a normal chain therewith:

THEOREM 11*e*. *In the elliptic Hermitian metric each hyperconic has one in-center whose polar does not meet the locus, and two out-centers whose polars meet it in chains. The hyperconic is its own reflection in each of the centers.*

There is a little more variety in the possible forms of hyperconic in the parabolic case. We have two possibilities

(*A*)
$$\begin{vmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{vmatrix} \neq 0.$$

Our hyperconic has the line equation

$$\Sigma A_{ij} u_i \bar{u}_j = 0:$$

we consider it at the same time as the form

$$u_1\bar{u}_1 + u_2\bar{u}_2 = 0.$$

The characteristic equation is

$$\begin{vmatrix} A_{00} - \rho & A_{01} & A_{02} \\ \bar{A}_{01} & A_{11} - \rho & A_{12} \\ \bar{A}_{02} & \bar{A}_{12} & A_{22} - \rho \end{vmatrix} = 0, \quad A_{00} \neq 0.$$

There will be at least one line other than (1, 0, 0) which has the same pole with regard to the two Hermitian forms. We call this and its pole (0, 1, 0) while the pole of the infinite line (1, 0, 0) with regard to the hyperconic shall be called the point (1, 0, 0). We are thus enabled to make exactly the same reductions as before, and reach the canonical form (14)

(*b*)
$$\begin{vmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{vmatrix} = A_{00} = 0.$$

The point $(0, a_{22}, -a_{12})$ has for its polar $x_0 = 0$ the infinite line. By a rotation of the plane we may force $(0, 1, 0)$ into this disagreeable rôle. There will be another point of the locus of such a nature that its tangent is perpendicular to the line connecting it with $(0, 1, 0)$ and this we take for $(1, 0, 0)$. We thus get the canonical forms

$$\begin{aligned} a_{01}x_0\bar{x}_1 + \bar{a}_{01}\bar{x}_0x_1 + A_2x_2\bar{x}_2 &= 0, \\ y\bar{y} &= \alpha x + \alpha\bar{x}. \end{aligned} \tag{15h}$$

Note that all points on this locus are equidistant from the "focus" $(\alpha/2, 0)$ and the directrix $x = -\alpha/2$.

THEOREM 11*p*. *In the parabolic Hermitian metric a hyperconic has either one center, or else it meets the infinite line in a single point. In this latter case it is the locus of points equidistant from a given point and a given line. This line, the directrix, is the locus of the reflection of the focus in a tangent. Mutually perpendicular lines through the focus are conjugate with regard to the hyperconic, and tangents from a point on the directrix are mutually perpendicular in pairs.*

The usual geometric proofs for the parabola are applicable here. Suppose that a hyperconic has an out-center at the point $x_h = 1$ $x_k = x_l = 0$. Its equation may then be written

$$\beta_hx_h\bar{x}_h + \beta_kx_k\bar{x}_k - \beta_lx_l\bar{x}_l = 0.$$

The tangents from the out-center, which we call "asymptotes" will generate the variety

$$\beta_kx_k\bar{x}_k - \beta_lx_l\bar{x}_l = 0.$$

We may write our hyperconic parametrically in the form:

$$x_h = \frac{e^{i\phi}}{\sqrt{\beta_h}}, \qquad x_k = \frac{\sinh Ae^{i\psi}}{\sqrt{\beta_k}}, \qquad x_l = \frac{\cosh Ae^{i\chi}}{\sqrt{\beta_l}}.$$

The distance from this point to the asymptote

$$\frac{e^{i\psi}}{\sqrt{\beta_e}}x_k - \frac{e^{i\chi}}{\sqrt{\beta_{12}}}x_l = 0$$

is

$$\frac{\sinh A - \cosh A}{\sqrt{P + Q\cosh^2 A + R\sinh^2 A}}$$

an expression which approaches zero as A becomes infinite. Exactly similar reasoning holds in the parabolic case

THEOREM 12. *If a point of a central hyperconic recede indefinitely from an out-center, its distance from the nearest asymptote through that center becomes infinitely small.*

We mentioned earlier that in the case of the non-central parabolic hyperconic, there was one point, called the focus, through which conjugate lines were mutually perpendicular. Let us see if there are corresponding points in the case of the central hyperconics. If there be any such point, the line connecting it with a center must be an axis, or else perpendicular to an axis through the given point. In any case, the point lies on an axis. Writing the hyperconic

$$\Sigma A_i x_i \bar{x}_i = 0, \tag{14}$$

we call our assumed focus $(0, y_k, y_l)$. An arbitrary line through this point will have the coördinates $(u_h, y_l, - y_k)$. The perpendicular thereto will be $(- (y_k \bar{y}_k + y_e \bar{y}_e), \bar{u}_h y_l, - \bar{u}_h y_k)$. The two will be conjugate if

$$\frac{- (y_k \bar{y}_k + y_l \bar{y}_l)}{A_h} + \frac{y_l \bar{y}_l}{A_k} + \frac{y_k \bar{y}_k}{A_l} = 0,$$

$$y_h = 0, \qquad y_k = \sqrt{A_l(A_h - A_k)}e^{i\theta}, \qquad y_l = \sqrt{A_k(A_l - A_h)}e^{i\phi}.$$

Of the three coefficients A_i one must be negative; we assume the other two positive, writing

$$\frac{y_k \bar{y}_k}{A_l(A_h - A_k)} = \frac{y_l \bar{y}_l}{A_k(A_l - A_h)} = 1.$$

We see that the negative coefficient can not be A_h, in fact A_h must be the numerically larger of the two positive coefficients. Every point so obtained shall be called a "focus," its polar being the directrix. We then find by a direct calculation,

THEOREM 13. *On one of the axes which does not connect two out-centers of a central hyperconic, there is a chain of foci. Conjugate lines through a focus are mutually perpendicular. In the elliptic Hermitian metric, the ratio of the sines of the distances of a point of a hyperconic from a focus and from the corresponding directrix is the same, not only for all foci, but for all points of the hyperconic. In the parabolic case it is the distances themselves which have a constant ratio.*

It is occasionally advantageous to write the equation of a hyperconic in a general symbolic form in the Clebsch-Aronhold notation. For instance, if the line equation be

$$u_a \bar{u}_{\bar{a}} = 0,$$

while the condition for perpendicularity is

$$u_\beta \bar{u}_{\bar{\beta}}' = \bar{u}_{\bar{\beta}} u_\beta' = 0,$$

and if (v) and (w) be two mutually perpendicular tangents to the hyperconic

$$v_\alpha \bar{v}_{\bar{\alpha}} = 0, \qquad w_\alpha \bar{w}_{\bar{\alpha}} = 0,$$

$$v_\beta \bar{w}_{\bar{\beta}} = \bar{v}_{\bar{\beta}} w_\beta = 0,$$

$$\begin{vmatrix} v_\alpha & v_\beta \\ w_\alpha & w_\beta \end{vmatrix} \cdot \begin{vmatrix} \bar{v}_{\bar{\alpha}} & \bar{v}_{\bar{\beta}} \\ \bar{w}_{\bar{\alpha}} & \bar{w}_{\bar{\beta}} \end{vmatrix} = 0,$$

$$|\alpha\beta x| \cdot |\bar{\alpha}\bar{\beta}\bar{x}| = 0.$$

THEOREM 14. *The locus of points whence tangents to a hyperconic are mutually perpendicular in pairs, is in the elliptic metric, another hyperconic with the same centers. In the parabolic case it is a hypercircle, or the directrix of a non-central hyperconic.*

We saw in theorem 7 that the usual trigonometric formulas hold for a triangle when, and only when, the altitudes are concurrent. It is not difficult to see, however, that the law of sines holds for every triangle. If then in the elliptic case, the vertices of a triangle be $A_1A_2A_3$ and the foot of the perpendicular from A_i upon the opposite side be A_i', we have

$$\sin(A_iA_i') = \sin(A_iA_j)\sin A_k = \sin(A_iA_k)\sin A_j,$$

$$\sin(A_iA_j)\sin(A_iA_k)\sin A_i = \sin(A_iA_i')\sin(A_jA_k) = \textit{const.}$$

We call this expression the "sine amplitude" of the triangle, and write it $\sin(A_1A_2A_3)$.* If the coördinates of the vertices be (y) (z) (t) we have

$$\sin(A_1A_2A_3) = \frac{\sqrt{|yzt|}\sqrt{|\bar{y}\bar{z}\bar{t}|}}{\sqrt{(y\bar{y})}\sqrt{(z\bar{z})}\sqrt{(t\bar{t})}}.$$

Notice also

$$\frac{\sin A_i}{\sin(A_jA_k)} = \frac{\sin(A_1A_2A_3)}{\sin(A_jA_k)\sin(A_kA_i)\sin(A_iA_j)}.$$

In the parabolic case we may take as double the measure of a triangle

$$(A_iA_j)(A_iA_k)\sin A_i = (A_iA_i')(A_jA_k).$$

In terms of non-homogeneous coördinates this is

$$\sqrt{\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix}} \sqrt{\begin{vmatrix} \bar{x} & \bar{y} & 1 \\ \bar{x}' & \bar{y}' & 1 \\ \bar{x}'' & \bar{y}'' & 1 \end{vmatrix}}.$$

We see, however, that a transversal through the vertex of a triangle will divide the triangle into two others whose measures add up to the measure of the given triangle only when the point of division belongs to the normal

* Conf. the Author's "Elements of Non-Euclidean Geometry," Oxford, 1909, p. 170.

chain of the two vertices collinear therewith. Hence we do not readily arrive at our measure by a double integration, and the whole subject seems to me of secondary interest.

4. **Differential formulas.** The fundamental differential expression which interests us is the squared distance element which is given in the elliptic case by the equation

$$ds^2 = \frac{(x\bar{x})(dxd\bar{x}) - (xd\bar{x})(\bar{x}dx)}{(x\bar{x})^2}. \tag{15e}$$

We have similarly the angular element

$$d\theta^2 = \frac{(u\bar{u})(dud\bar{u}) - (ud\bar{u})(\bar{u}du)}{(u\bar{u})^2}.$$

It is frequently better to drop the homogeneous point and line coordinates in differential work. Thus, assuming that a line does not pass through the point (1, 0, 0), we write its equation

$$ux + vy + 1 = 0,$$

and have for our differential forms

$$ds^2 = \frac{dxd\bar{x} + dyd\bar{y} + (xdy - ydx)(\bar{x}d\bar{y} - \bar{y}d\bar{x})}{(x\bar{x} + y\bar{y} + 1)^2}, \tag{16e}$$

$$d\theta^2 = \frac{dud\bar{u} + dvd\bar{v} + (udv - vdu)(\bar{u}d\bar{v} - \bar{v}d\bar{u})}{(u\bar{u} + v\bar{v} + 1)^2}. \tag{17e}$$

In the parabolic case we have the analogous but simpler expressions

$$dx^2 = dxd\bar{x} + dyd\bar{y}, \tag{16h}$$

$$d\theta^2 = \frac{(udv - vdu)(\bar{u}d\bar{v} - \bar{v}d\bar{u})}{(u\bar{u} + v\bar{v})^2}. \tag{17h}$$

Suppose that we have given a curve

$$y = y(x), \qquad \bar{y} = \bar{y}(\bar{x}).$$

The equation of the tangent is

$$\frac{y'}{y - xy'}X - \frac{1}{y - xy'}Y + 1 = 0,$$

$$du = \frac{yy''}{(y - xy')^2}dx, \qquad dv = \frac{-\,xy''}{(y - xy')^2}dx,$$

$$d\bar{u} = \frac{\bar{y}\bar{y}''}{(\bar{y} - \bar{x}\bar{y}^1)^2}d\bar{x}, \qquad d\bar{v} = \frac{-\,\bar{x}\bar{y}''}{(\bar{y} - \bar{x}\bar{y}^1)^2}d\bar{x},$$

$$ds^2 = \frac{1 + y'\bar{y}' + (xy' - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\,dx d\bar{x}:$$

$$d\theta^2 = \frac{(x\bar{x} + y\bar{y} + 1)y''\bar{y}''dx d\bar{x}}{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]^2}.$$

We have, thus as an expression for the curvature

$$\frac{1}{k} = \frac{d\theta}{ds} = \frac{\sqrt{y''\bar{y}''}}{\left[\dfrac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^{\frac{3}{2}}}. \tag{18e}$$

Suppose, secondly we consider a complex surface where the distance element is

$$ds^2 = 0dx^2 + 2Fdx d\bar{x} + 0d\bar{x}^2 = \frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\,dx d\bar{x}.$$

We may write this a little more simply in the form

$$2F = \frac{u\bar{u} + v\bar{v} + 1}{v\bar{v}(x\bar{x} + y\bar{y} + 1)^2}.$$

We now look for the Gaussian curvature of this surface

$$\frac{\partial^2 \log F}{\partial^2 x d\bar{x}} = \left[\frac{\partial^2 \log (u\bar{u} + v\bar{v} + 1)}{\partial x \partial \bar{x}} - 2\frac{\partial^2 \log (x\bar{x} + y\bar{y} + 1)}{\partial x \partial \bar{x}}\right]$$

$$= \left[\frac{u'\bar{u}' + v'\bar{v}' + (uv' - vu')(\bar{u}\bar{v}' - \bar{v}\bar{u}')}{(u\bar{u} + v\bar{v} + 1)^2}\right.$$

$$\left. - 2\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{y}\bar{x}')}{(x\bar{x} + y\bar{y} + 1)^2}\right]$$

$$= \left[\frac{(1 + x\bar{x} + y\bar{y})y''\bar{y}''}{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]^2}\right.$$

$$\left. - 2\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\right],$$

$$\frac{1}{K} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial x \partial \bar{x}} = -\frac{2}{k^2} - 4. \tag{19e}$$

THEOREM 14. *The Gaussian curvature of a surface having the same distance element as a given curve is, in the elliptic case, 4 less than minus two times the square of the curvature of the given curve. In the parabolic case the difference of 4 between the two expressions is lacking.*

We next look for a curve of constant curvature

$$\frac{y''\bar{y}''}{\left[\dfrac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^3} = C.$$

Assuming C not to be zero, we treat x and $\bar{x}$ as independent variables and differentiate the logarithms of both sides successively to x and to $\bar{x}$

$$\frac{y''\bar{y}''}{\left[\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^3} = -2.$$

This is an absurd equation, hence we must have $C = 0$.

THEOREM 16. *The only curves of constant curvature in Hermitian metrics are straight lines.*

Let us look for a "geodesic thread" on a given curve, i.e., a system of points depending upon one real parameter which gives the shortest path between two of its members, the normal chain is a good example. We assume as before that y is a known function of x and $\bar{y}$ the conjugate imaginary function of $\bar{x}$. Let us find two functions $x = x(t)$ $\bar{x} = \bar{x}(t)$ which will minimize the integral

$$\int_a^b \sqrt{\frac{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]x'\bar{x}'dt}{(x\bar{x} + y\bar{y} + 1)^2}}.$$

We call this

$$\int_a^b R(x\bar{x}x'\bar{x}')dt = \int_a^b \sqrt{F(x, \bar{x}) \cdot x'\bar{x}'}dt.$$

To minimize this we may treat x and $\bar{x}$ as independent

$$\frac{\partial R}{\partial x} = \frac{d}{dt}\left(\frac{\partial R}{\partial x'}\right), \qquad \frac{\partial R}{\partial \bar{x}} = \frac{d}{dt}\left(\frac{\partial R}{\partial x'}\right).$$

These reduce to the single equation

$$x'\frac{\partial F}{\partial x} - \bar{x}'\frac{\partial F}{\partial \bar{x}} = \frac{F}{x'\bar{x}'}(x'\bar{x}'' - \bar{x}'x'').$$

Let

$$\phi = \log F, \qquad x'\frac{\partial \phi}{\partial x} - \bar{x}'\frac{\partial \phi}{\partial \bar{x}} = \frac{x'\bar{x}'' - \bar{x}'x''}{x'\bar{x}'}.$$

Let

$$x = U + iV, \qquad \bar{x} = U - iV,$$

$$x' = U' + iV', \qquad \bar{x}' = U' - iV',$$

$$V'\frac{\partial \phi}{\partial U} - U'\frac{\partial \phi}{\partial V} = \frac{2(V'U'' - U'V'')}{U'^2 + V'^2}$$

$$\frac{dV}{dU} = \frac{V'}{U'}, \qquad V' = U'\frac{dV}{dU},$$

$$U'^3\frac{d^2V}{dU^2} = U'V'' - V'U'',$$

$$\frac{d^2V}{dU^2} + \frac{1}{2}\left[1 + \left(\frac{dV}{dU}\right)^2\right]\left[\frac{dV}{dU}\frac{\partial \phi}{\partial U} - \frac{\partial \phi}{\partial V}\right] = 0.$$

In the case of the line $y = 0$,

$$F = \frac{1}{(x\bar{x} + 1)^2},$$

$$(x\bar{x} + 1)(x'\bar{x}'' - \bar{x}'x'') + 2x'\bar{x}'(\bar{x}x' - x\bar{x}') = 0.$$

Let

$$x = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \bar{x} = \frac{\bar{\alpha} t + \bar{\beta}}{\bar{\gamma} t + \bar{\delta}},$$

$$\alpha\bar{\beta} + \gamma\bar{\delta} = \bar{\alpha}\beta + \bar{\gamma}\delta,$$

a normal chain.

CAMBRIDGE, MASS.

ON THE EXPANSION OF CERTAIN ANALYTIC FUNCTIONS IN SERIES.*

By R. D. Carmichael.

Introduction. Let ρ be a positive constant and let $g(x)$ be a function of x with the asymptotic representation

$$g(x) \sim x^{\mu-\rho x} e^{\alpha+\beta x}\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right) \tag{1}$$

valid in a sector V including in its interior the positive axis of reals and every point of the imaginary axis. Moreover let $g(x)$ be analytic at every point x in V for which $|x|$ is greater than a given constant γ. By means of this function $g(x)$ we set up the class of series which have the form

$$S(x) = \sum_{k=0}^{\infty} c_k \frac{g(x+k)}{g(x)}, \tag{2}$$

where $c_0, c_1, c_2, \cdots$ are constants. These (and other more general) series I have already treated in several papers.† The object of this note is to derive certain necessary and sufficient conditions for the representation of functions in the form of series $S(x + a)$ where a is a suitable constant depending on the function to be represented.

1. **Representation of a function of a certain class by means of a suitable integral along a straight line.** Let $f(x)$ be a function of x which is analytic at every finite point to the right of a line L_1 parallel to the axis of imaginaries. Let L denote a line parallel to L_1 and to its right; and let us think of L as having the direction which leads from a point with negative imaginary part to one with positive imaginary part. On L let $f(x)$ be bounded by the inequality

$$|f(x)| < \frac{N}{|x|^{\epsilon}}, \tag{3}$$

when $|x|$ is greater than a sufficiently large positive constant X, where ϵ and N are given positive constants. We shall show that *such a function*

* Presented to the American Mathematical Society, April, 1920.

† These memoirs will be referred to by the numbers in the following list: I. Trans. Am. Math. Soc., vol. 17 (1916): 207–232. II. Bulletin Am. Math. Soc., vol. 23 (1917): 407–425. III. American Journal of Mathematics, vol. 39 (1917): 385–403. IV. American Journal of Mathematics, vol. 40 (1918): 113–126.

$f(x)$ may be written in the form of the integral

$$f(x) = \frac{1}{2\pi i}\int_L \frac{f(z)dz}{x - z}, \tag{4}$$

where the integral is taken along L in the direction already indicated and is valid for every point x to the right of L. For this purpose consider the semicircle of radius t about a point α on L as center and lying to the right of L. Let Δ_t denote the region inclosed by the semicircle C_t and its diameter D_t. Let x be any point to the right of L and let t be so large that x is in the interior of Δ_t. Then by the Cauchy integral formula we have

$$f(x) = \frac{1}{2\pi i}\int_{D_t} \frac{f(z)dz}{z - x} + \frac{1}{2\pi i}\int_{C_t} \frac{f(z)dz}{z - x},$$

where the integrals are taken along D_t and C_t in a positive sense with reference to the inclosed area Δ_t.

Now let the radius t of C_t become infinite. The second integral in the last equation approaches zero, as one sees readily by taking the absolute value of the integrand and employing inequality (3). The first integral in the last equation approaches the integral in (4). Hence $f(x)$ can be written in the form given in (4) and the representation is valid for every point x to the right of L.

2. **The convergence of certain asymptotic representations with respect to** $g(x)$. Let $f(x)$ be a function which is analytic at every (finite) point to the right of a line L_1 parallel to the axes of imaginaries. Denote by L a line to the right of L_1 and parallel to it. Let a constant a be chosen so that for x on L the real part $R(x + a)$ of $x + a$ is greater than a non-negative integer r and so that $g(x + a + 1)/g(x + a)$ is analytic and different from zero at every finite point to the right of the line L_1. Then for every positive integer n the function $g(x + a + n)/g(x + a)$ has the last-mentioned property since

$$\frac{g(x + a + n)}{g(x + a)} = \frac{g(x + a + 1)}{g(x + a)} \cdot \frac{g(x + a + 2)}{g(x + a + 1)} \cdots \frac{g(x + a + n)}{g(x + a + n - 1)}. \tag{5}$$

Let $f(x)$ be written in the form

$$f(x) = \sum_{k=0}^{n} c_k \frac{g(x + a + k)}{g(x + a)} + R_n(x)\frac{g(x + a + n + 1)}{g(x + a)}. \tag{6}$$

For every x on L and for every positive integer n let $R_n(x)$ satisfy the inequality

$$|R_n(x)| \leqq (n!)^\rho e^{nR(\rho - \beta)} n^{r\rho + R(\frac{1}{2}\rho - n)} \epsilon_n, \tag{7}$$

where ϵ_n approaches zero with $1/n$. Moreover, for x to the right of L, let $f(x)$ be such that the function $f(x) - c_0$ is bounded by the inequality

$$|f(x) - c_0| < N|x|^{-\epsilon}$$

when $|x|$ is greater than a sufficiently large positive quantity X, where ϵ and N are positive constants. Then we shall show that $f(x)$ *may be represented by the series*

$$f(x) = \sum_{k=0}^{\infty} c_k \frac{g(x+a+k)}{g(x+a)} \tag{8}$$

and that this series converges for every x to the right of L.

From the asymptotic character of $g(x)$ it follows readily that $(x+a)^{s\rho} g(x+a+s)/g(x+a)$, and hence $x^{s\rho} g(x+a+s)/g(x+a)$, approaches a finite non-zero limit as $x + a$ approaches infinity in V. Hence when x is to the right of L it is easy to see that the remainder term in (6) may be represented in the form of an integral in accordance with the result in § 1. Hence we have

$$f(x) = \sum_{k=0}^{n} c_k \frac{g(x+a+k)}{g(x+a)} + \frac{1}{2\pi i}\int_L \frac{R_n(z)}{x-z}\,\frac{g(z+a+n+1)}{g(z+a)}\,dz, \tag{9}$$

where x is to the right of L and the integration along L is taken in the direction indicated in § 1 and x is to the right of L.

In order to prove that $f(x)$ has the convergent representation (7) it is now necessary and sufficient to show that the integral in the last equation approaches the value zero as n becomes infinite. But this integral is not greater in absolute value than the integral

$$\int_L \left|\frac{R_n(z)}{z-x}\right| \cdot \left|\frac{g(z+a+n+1)}{g(z+a)}\right| \cdot |dz|.$$

In order to find a function of n which dominates the latter integral we employ the asymptotic relation

$$g(x)\{\Gamma(x)\}^{\rho} \sim x^{\mu-\rho x} e^{\alpha+\beta x}\left(1 + \frac{a_1}{x} + \cdots\right)\cdot x^{-\frac{1}{2}+x} e^{-x}\sqrt{2\pi}\left(1+\frac{b_1}{x}+\cdots\right)^{\rho}$$

$$\sim x^{\mu-\rho/2} e^{\alpha+(\beta-\rho)x}(\sqrt{2\pi})^{\rho}\left(1+\frac{c_1}{x}+\cdots\right).$$

From this we have

$$\frac{g(x+n+1)}{g(x)}\left\{\frac{\Gamma(x+n+1)}{\Gamma(x)}\right\}^{\rho} = \left(1+\frac{n}{x}\right)^{\mu-\rho/2} e^{(\beta-\rho)n} S_{(x,\,n)},$$

where $S(x, n)$ approaches 1 when n becomes infinite or x becomes infinite in V or both of these conditions are realized simultaneously. Hence

for z on L constants M_1 and M_2 exist such that

$$(10)\quad \left|\frac{g(z+a+n+1)}{g(z+a)}\right| < M_1 \left|(z+a)(z+a+1)\cdots(z+a+n)\right|^{-\rho} e^{nR(\beta-\rho)} n^{R(\mu-\frac{1}{2}\rho)} < M_2 |z+a|^{-\rho}(n!)^{-\rho} e^{nR(\beta-\rho)} n^{R(\mu-\frac{1}{2}\rho)-r\rho}.$$

Employing this inequality and (7) we see that a constant M exists such that the last-written integral has a value less than $M\epsilon_n$. Hence it approaches zero with $1/n$. Hence from (9) we have (8), a relation which is valid if x is to the right of L, as was to be proved.

Now let L_2 be a line to the right of L and parallel to it. Since $g(x+a+n)/g(x+a)$ is analytic and different from zero at every point to the right of L_1, it follows that there is no exceptional point for the series in (8) and no limit point of such points to the right of L. Therefore, from theorem III of memoir III it follows that the series in (8) is uniformly convergent for x in the region consisting of L_2 and the half-plane to its right (exclusive of the point infinity).

We are thus led to the following theorem:

Let $f(x)$ be a function which is analytic at every (finite) point to the right of a line L_1 parallel to the axis of imaginaries and suppose that constants c_0 and $\epsilon(\epsilon > 0)$ exist such that $|x|^{\epsilon}\cdot|f(x) - c_0|$ is bounded as x approaches infinity to the right of L_1. Denote by L a line to the right of L_1 and parallel to it. Let a constant a be chosen so that for X on L $R(x+a)$ is greater than a non-negative integer r and so that $g(x+a+1)/g(x+a)$ is analytic and different from zero at every finite point to the right of L_1. If $f(x)$ is written in the form (6) then let $R_n(x)$, for every x on L and for every positive integer n, satisfy inequality (7). Then $f(x)$ may be represented by the series in (8) and this series converges for every finite x to the right of L. Moreover, it converges uniformly in the region consisting of a line to the right of and parallel to L and the half-plane to the right of such line (exclusive of the point infinity).

3. **Necessary and sufficient conditions for the representation of a function $f(x)$ in the form of a series $S(x+a)$.** Let $F(x)$ be a function represented in the form of a series $S(x)$,

$$(11)\qquad F(x) = \sum_{k=0}^{\infty} \gamma_k \frac{g(x+k)}{g(x)},$$

converging for some value of x which is non-exceptional for this series and hence in a half-plane $R(x) > l$, where the line $R(x) = l$ is the boundary of the region of convergence.

From theorem I of memoir IV it follows that a constant S_1 exists such that

$$|\gamma_n g(n)| < n^{s_1}, \quad n \geqq 2.$$

Now we have with respect to n the asymptotic formula

$$g(n)\{\Gamma(n)\}^{\rho} \sim e^{n(\beta-\rho)} n^{\mu-\frac{1}{2}\rho} e^{\alpha} (\sqrt{2\pi})^{\rho} \left(1 + \frac{c_1}{n} + \cdots\right).$$

Hence a constant s exists such that

$$(12) \qquad |\gamma_{n+1}| < (n!)^{\rho} e^{nR(\rho-\beta)} n^{s}, \quad n \geqq 2.$$

Let us now write $F(x)$ in the form

$$F(x) = \sum_{k=0}^{n} \gamma_k \frac{g(x+k)}{g(x)} + \bar{R}_n(x) \frac{g(x+n+1)}{g(x)}.$$

Then we have

$$(13) \qquad \bar{R}_n(x) = \sum_{k=n+1}^{\infty} \gamma_k \frac{g(x+k)}{g(x+n+1)}$$

provided that x is confined to a portion of the region of convergence of the series in (11) for which $g(x+n+1)/g(x)$ is different from zero. If x is on a line L parallel to the axis of imaginaries and sufficiently far to its right, we may employ relation (10) with z replaced by x, a by $n+1$, and n by $k-n-2$, to conclude to the existence of a constant M_3 (independent of x and n) such that

$$\left|\frac{g(x+k)}{g(x+n-1)}\right| < M_3 |(x+n+1)\cdots(x+k-1)|^{-\rho} e^{(k-n-2)R(\beta-\rho)} (k-n-2)^{R(\mu-\frac{1}{2}\rho)},$$

where $k > n+2$. Hence we have

$$\left|\gamma_k \frac{g(x+k)}{g(x+n+1)}\right|$$

$$< M_3 \left|\frac{(k-1)!}{(x+n-1)\cdots(x+k-1)}\right|^{\rho} e^{(n+1)R(\rho-\beta)} (k-n-2)^{R(\mu-\frac{1}{2}\rho)} n^{s}$$

$$< M_3 e^{R(\rho-\beta)} (n!)^{\rho} e^{nR(\rho-\beta)} n^{s} \left|\frac{(n+1)\cdots(k-1)}{(x+n+1)\cdots(x+k-1)}\right|^{\rho} (k-n-2)^{R(\mu-\frac{1}{2}\rho)}$$

$$< M_4 (n!)^{\rho} e^{nR(\rho-\beta)} n^{s+t\rho} \left|\frac{(n+t+1)\cdots(k-1)}{(x+n+1)\cdots(x+n+k-t-1)}\right|^{\rho}$$

$$\cdot k^{-\rho t} (k-n-2)^{R(\mu-\frac{1}{2}\rho)},$$

where t is a positive integer such that $\rho t > R(\mu - \frac{1}{2}\rho) + 2$, $k > n + t + 2$, and M_4 is a suitable constant. Hence if the line L is sufficiently far to

the right, then for x on L we have

$$\left|\gamma_k \frac{g(x+k)}{g(x+n+1)}\right| < M_5 (n!)^{\rho} e^{nK(\rho-\beta)} n^{s+t\rho} k^{-2}, \qquad k > n+t+2,$$

where M_5 is a suitable quantity independent of x and n.

Employing the last inequality and relations (12) and (13) we find that for x on L we have

$$|\bar{R}_n(x)| < \sum_{k=n+1}^{n+t+2} [(k-1)!]^{\rho} e^{(k-1)K(\rho-\beta)} (k-1)^s \left|\frac{g(x+k)}{g(x+n+1)}\right|$$
$$+ \sum_{k=n+t+3}^{\infty} M_5 (n!)^{\rho} e^{nK(\rho-\beta)} n^{s+t\rho} \frac{1}{k^2}.$$

Hence a quantity M_6, independent of x and n, exists such that for x on L we have

$$|\bar{R}_n(x)| < M_6 (n!)^{\rho} e^{nK(\rho-\beta)} n^{s+t\rho}.$$

Therefore an r exists such that $\bar{R}_n(x)$ satisfies the condition imposed on $R_n(x)$ in inequality (7), provided that the line L has been taken sufficiently far to the right.

It may now be readily shown that for the given function $F(x)$ lines L_1 and L exist such that $F(x)$ has the same properties with respect to them and to a constant a (in this case zero) as are specified for $f(x)$ in the hypothesis of the theorem of § 2. Analyticity to the right of some line L_1 parallel to the axis of imaginaries follows from the last result in § 3 of memoir I. From theorem I of memoir III it follows that $|x|^{\rho} \cdot |F(x) - \gamma_0|$ is bounded for x approaching infinity in a half-plane to the right of a suitable line L_1. We have just demonstrated the appropriate property of the factor $\bar{R}_n(x)$ in the remainder term. Hence the conditions given in the theorem of § 2 and there found to be sufficient for the representation of a function $f(x)$ in the form of a series $S(x+a)$ are also properties of every function so represented. *Thus we have necessary and sufficient conditions for the representation of functions in the form of series $S(x+a)$.*

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NOTES ON THE CYCLIC QUADRILATERAL.

By Frank V. Morley.

A triangle gives rise to certain well-defined and unique points, such as the centroid, symmedian point, etc., and these points may be considered as attached to the triangle. Four vertices will give rise to four such triangles, successively obtained by omitting each vertex in turn, and there is a natural inquiry into the behavior of the four sets of attached points. In general the relations are not always simple; but there are certain pretty simplifications when the four vertices are all on a circle.

If we call four points on a circle α, β, γ, δ, the four triangles will be $\alpha\beta\gamma$, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$. By definition they all have the same circumcenter, o. This suggests a treatment by vector analysis, in which we take o as origin and consider the circle as a unit circle or base circle. Then α, β, γ, δ are orthogonal numbers, or *turns*, upon this circle.

It may be convenient to introduce the symmetric functions of α, β, γ, δ. For three points α, β, γ these are

$$s_1 = \alpha + \beta + \gamma, \qquad s_2 = \alpha\beta + \beta\gamma + \gamma\alpha, \qquad s_3 = \alpha\beta\gamma.$$

Similarly for four points the symmetric functions are the sums taken one, two, three, and four at a time. The context makes it clear as to whether the symmetric functions are for three or four points.

Besides the single points attached to the triangle, of which we have cited the centroid and symmedian point as examples, there are certain pairs of points. The most interesting of these are the Hessian points and the equiangular or Fermat points. These will give rise to simple configurations when considered for the four triangles which make up an inscribed quadrilateral. And finally the particular set of four points formed by the incenters of a triangle gives rise to a recently studied rectangular net when considered for the four triangles of an inscribed quadrilateral.

Although all of the proofs might be thrown into the notation of vector analysis, it will be found more convenient in some cases to indicate other methods. It may be said that the facts were largely suggested by the writer, while the methods of proof were generally intimated by his father, Professor Morley.

1. **Centroids.** The centroid of the triangle α, β, γ may be called g_δ,

and is by definition

$$x = \frac{\alpha + \beta + \gamma}{3}.$$

This may be written as

$$3x = \alpha + \beta + \gamma + \delta - t$$

or

$$3x = s_1 - t,$$

where t is a variable turn traveling round the base circle. The equation is now *symmetrized.* For a varying t it represents a circle, and when t picks up the particular values α, β, γ, δ, x is in succession the point g_α, g_β, g_γ, g_δ. Hence *the four centroids are on a circle,* with center $s_1/3$ and radius $1/3$.

Moreover,

$$3g_\alpha = s_1 - \alpha, \qquad 3g_\beta = s_1 - \beta,$$

so that

$$3(g_\alpha - g_\beta) = -(\alpha - \beta).$$

From this we see that $g_\alpha g_\beta$ is parallel to $\alpha\beta$ and one-third of its length. Hence *the four centroids form a cyclic quadrilateral similar to the four vertices and parallel in situation, but of one-third the size.*

2. **Orthocenters.** The orthocenter of a triangle, h_δ, is given by

$$x = \alpha + \beta + \gamma.$$

This may be symmetrized and written

$$x = s_1 - t.$$

This is again a circle, and when t picks up the value α, β, γ, δ, x is in succession h_α, h_β, h_γ, h_δ. Hence *the four orthocenters are on a circle,* with center s_1 and radius 1. Moreover,

$$h_\alpha - h_\beta = -(\alpha - \beta),$$

so that *the four orthocenters form a cyclic quadrilateral equal to the quadrilateral formed by the four vertices, and parallel in situation.*

3. **Centers of nine-point circles.** The center of the nine-point circle, n_δ, is found to be

$$x = \frac{\alpha + \beta + \gamma}{2}.$$

This may be symmetrized and written for the four triangles as

$$2x = s_1 - t.$$

It follows that *the centers of the four nine-point circles form a cyclic quadrilateral similar to the quadrilateral formed by the four vertices, and parallel in situation, but of half the size.*

4. **Symmedian points.** The symmedian point, k, of a triangle (here the symmetric functions are for three variables) is found by direct calculation to be

$$x = \frac{6s_1s_3 - 2s_2}{9s_3 - s_1s_2}.$$

It is not convenient to symmetrize this expression, and our previous treatment breaks down.

It is then advisable to find out how the symmedian point appears in barycentric coördinates. Since we are to deal with a quadrilateral, symmetry will be gained by choosing the diagonal triangle for reference. Then the four vertices will have coördinates

$$\begin{array}{llll} \alpha: & a_0 & a_1 & a_2 \\ \beta: & -a_0 & a_1 & a_2 \\ \gamma: & a_0 & -a_1 & a_2 \\ \delta: & a_0 & a_1 & -a_2. \end{array}$$

These four points are to lie on the circle apolar to the reference triangle,

$$c_0x_0^2 + c_1x_1^2 + c_2x_2^2 = 0, \tag{1}$$

where the c's are the cotangents of the angles of the reference triangle.

Let us calculate the symmedian point of $\beta\gamma\delta$. The tangent to (1) at β is

$$-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2 = 0,$$

and at γ is

$$c_0a_0x_0 - c_1a_1x_1 + c_2a_2x_2 = 0.$$

Any line through their intersection is

$$-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2 + \lambda c_0a_0x_0 - \lambda c_1a_1x_1 + \lambda c_2a_2x_2 = 0.$$

This line passes through δ if λ is so chosen that

$$-c_0a_0^2 + c_1a_1^2 - c_2a_2^2 + \lambda c_0a_0^2 - \lambda c_1a_1^2 - \lambda c_2a_2^2 = 0.$$

By virtue of (1) this reduces to

$$c_1a_1^2 + \lambda c_0a_0^2 = 0.$$

Substituting this value of λ and dividing by $c_0a_0^2c_1a_1^2c_2a_2^2$, we have for the symmedian line through δ the equation

$$\frac{-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2}{c_1a_1^2c_2a_2^2} = \frac{c_0a_0x_0 - c_1a_1x_1 + c_2a_2x_2}{c_2a_2^2c_0a_0^2} = m,$$

where m is unaltered by interchange of letters, as appears immediately

when the letters are permuted cyclically to form the equations of the symmedian lines through β and γ. By adding numerators and denominators in the equation written,

$$\frac{2c_2a_2x_2}{c_2a_2^2(c_0a_0^2 + c_1a_1^2)} = m$$

and again using the reduction (1),

$$x = \frac{-m}{2}c_2a_2^3.$$

The symmedian point is then found to have as coördinates simply

$$k_0 = c_0a_0^3, \qquad k_1 = c_1a_1^3, \qquad k_2 = c_2a_2^3.$$

The four symmedian points derived from the four triangles are therefore

$$\begin{array}{lccc} k \ : & k_0 & k_1 & k_2 \\ k_\beta : & -k_0 & k_1 & k_2 \\ k_\gamma : & k_0 & -k_1 & k_2 \\ k_\delta : & k_0 & k_1 & -k_2. \end{array}$$

Comparison with the coördinates of α, β, γ, δ shows that *the four symmedian points have the same diagonal triangle as the four vertices.*

This may also be seen from the theorem that *there is a particular projection which sends a circle with an inscribed quadrilateral into a circle with an inscribed rectangle.* This is proved by Professor Morley as follows: Let v be the exterior diagonal of the four points α, β, γ, δ on a circle C in a plane P. Take a sphere on α, β, γ, δ. Draw either tangent plane from v to the sphere, and let N be the point of contact. Take any plane P' parallel to this tangent plane. When we project from N the circle C will become a circle C' in P', and also the four points α, β, γ, δ will become the vertices of a parallelogram, since the third or exterior diagonal has gone to infinity. Thus the projection of the circle with the inscribed quadrilateral α, β, γ, δ is a circle with an inscribed parallelogram; i.e., a rectangle.

When the circle C is projected into a circle C', the tangents of C are projected into the tangents of C', and therefore the symmedian point of a triangle inscribed in C goes into the symmedian point of the triangle in C'. When we project four points on C into a rectangle on C', the four symmedian points become the four symmedian points of the triangles formed from the rectangle. But these form a concentric rectangle, or a rectangle having the same diagonal triangle. Hence, by projecting back, the original symmedian points have the same diagonal triangle as the four concyclic vertices.

Returning to the barycentric expressions, we derived a simple cubic transformation which sends the inscribed quadrilateral into its symmedian points, namely

$$k_i = c_i a_i^3.$$

In this the a's may be eliminated, with the resulting locus for k,

$$c_0^{1/3}x_0^{2/3} + c_1^{1/3}x_1^{2/3} + c_2^{1/3}x_2^{2/3} = 0. \tag{2}$$

The relation of this locus and the circle

$$c_0x_0^2 + c_1x_1^2 + c_2x_2^2 = 0, \tag{1}$$

savors strongly of the familiar case in rectangular coördinates of the astroid

$$X^{2/3} + Y^{2/3} = A^{2/3}, \tag{3}$$

and its circumcircle

$$X^2 + Y^2 = A^2. \tag{4}$$

In fact the symmedian locus (2) is the particular projection of the astroid in which the circle (4) goes into the circle (1).

It is then clear that the four symmedian points of an inscribed quadrilateral are on a six-cusped curve whose cusps are on the sides of the diagonal triangle. This curve, together with the diagonal triangle, affords a unique construction for the symmedian quadrilateral when one symmedian point is given.

There is another way in which the peculiarity of the symmedian quadrilateral may be stated. Any four points a_i set up a pencil of conics. The polar of a point x with respect to the pencil is a pencil of lines through y. Hence x and y are in a quadratic Cremona involution

$$x_iy_i = a_i^2.$$

This is a transformation over the plane. In particular, it sends the orthocenter of the diagonal triangle, namely

$$y_0 = c_1c_2, \qquad y_1 = c_2c_0, \qquad y_2 = c_0c_1, \tag{5}$$

into

$$x_0 = c_0a_0^2, \qquad x_1 = c_1a_1^2, \qquad x_2 = c_2a_2^2. \tag{6}$$

When the four points a_i are on the circle (2), (6) is at infinity, and the transformation sends the orthocenter to infinity. But when the four points setting up the involution are k_i, (6) becomes

$$x_0 = (c_0a_0^2)^3, \qquad x_1 = (c_1a_1^2)^3, \qquad x_2 = (c_2a_2^2)^3;$$

a point on the cubic

$$x_0^{1/3} + x_1^{1/3} + x_2^{1/3} = 0. \tag{7}$$

Hence the symmedian points are such that their involution sends the orthocenter of their diagonal triangle into a point on the cubic (7).

5. **Hessian points.** A Hessian point may be defined as a point whose images in the sides of a triangle form an equilateral triangle. Reverting to vector analysis, this will lead to the expression*

$$h_\delta = -\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\alpha + \omega^2\beta + \omega\gamma}.$$

There will be two such points for any triangle; the other point of the pair, h_δ', is found by interchanging ω and ω^2, where ω and ω^2 are the complex cube roots of 1.

The above expression is for the triangle formed by omitting δ. The corresponding Hessian, h_α, for the triangle formed by omitting α will be found by cyclically permuting the letters,

$$h_\alpha = -\frac{\gamma\delta + \omega^2\delta\beta + \omega\beta\gamma}{\beta + \omega^2\gamma + \omega\delta}.$$

But if we multiply this second expression by ω^2 above and below, it will be seen to be the same as the first, except that α is replaced by δ. Hence $h_\delta - h_\alpha$ will have a factor $(\delta - \alpha)$, and may be written

$$h_\delta - h_\alpha = \frac{(\delta - \alpha)[\beta\gamma(1 - \omega + \omega^2) + \omega(\beta^2 + \gamma^2)]}{(\alpha + \omega^2\beta + \omega\gamma)(\beta + \omega^2\gamma + \omega\delta)}$$

$$= \frac{\omega(\delta - \alpha)(\beta - \gamma)^2}{(\alpha + \omega^2\beta + \omega\gamma)(\beta + \omega^2\gamma + \omega\delta)}.$$

In a similar way

$$h_\beta - h_\gamma = \frac{\omega(\beta - \gamma)(\delta - \alpha)^2}{(\gamma + \omega^2\delta + \omega\alpha)(\delta + \omega^2\alpha + \omega\beta)},$$

and similarly for the other differences. The double ratio for the Hessians h_α, h_β, h_γ, h_δ, is then

$$\frac{(h_\delta - h_\alpha)(h_\beta - h_\gamma)}{(h_\alpha - h_\beta)(h_\gamma - h_\delta)} = \frac{(\delta - \alpha)^3(\beta - \gamma)^3}{(\alpha - \beta)^3(\gamma - \delta)^3},$$

and the double ratio for the four points α, β, γ, δ is

$$\frac{(\delta - \alpha)(\beta - \gamma)}{(\alpha - \beta)(\gamma - \delta)}.$$

Since α, β, γ, δ are on a circle, this last double ratio is real; hence the double ratio for the Hessians, being the cube of this, is also real. A similar result will be obtained for the set of points h_α', h_β', h_γ', h_δ'. Therefore *each set of Hessian points is on a circle.*

* Harkness and Morley, Theory of Functions, p. 26.

6. **Equiangular points.** An old problem, attributed to Fermat, is to find a point the sum of whose distances from three vertices, say α, β, γ, is a minimum. At such a point the angles subtended by $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ will be equal or supplementary. There are in fact two points, familiar as the intersections of the lines joining α, β, γ to the vertices of equilateral triangles described, all outwards or all inwards, on $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$. Another definition would show the points to be the isogonal conjugates of the Hessian pair. They are variously known as Fermat points, isogonal centers,* or equiangular points.

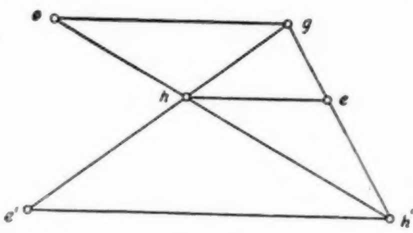

FIG. 1.

To handle the equiangular points by vector analysis, a convenient starting point is Figure 1, taken from a paper by Professor Morley.† Here o is the circumcenter of any triangle, g the centroid, h, h' the Hessian pair, and e, e' the equiangular points. Immediately it is seen that

$$e - g = (h' - g)\frac{h}{h'}.$$

Substituting the previous values for g, h, and h', we have

$$3e_\delta = \alpha + \beta + \gamma - (\alpha + \omega^2\beta + \omega\gamma)\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}.$$

Let us write

$$3(e_\delta - \alpha) = \frac{(\beta+\gamma-2\alpha)(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta) - (\alpha+\omega^2\beta+\omega\gamma)(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)}{\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta}.$$

If in this expression α were equal to β, the numerator would vanish, which is convenient to note geometrically; so that $(\alpha - \beta)$ is a factor. Similarly $(\alpha - \gamma)$ is a factor, and the expression may be written

$$3(e_\delta - \alpha) = \frac{\omega(\omega - \omega^2)(\omega\gamma - \beta)(\alpha - \beta)(\alpha - \gamma)}{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}.$$

We may now permute the letters and write (since $\omega - \omega^2 = i\sqrt{3}$),

$$i\sqrt{3}(e_\delta - \beta) = \frac{\omega(\gamma - \omega\alpha)(\beta - \gamma)(\beta - \alpha)}{\gamma\alpha + \omega\alpha\beta + \omega^2\beta\gamma},$$

and also

$$i\sqrt{3}(e_\alpha - \beta) = \frac{\omega(\gamma - \omega\delta)(\beta - \gamma)(\beta - \delta)}{\gamma\delta + \omega\delta\beta + \omega^2\beta\gamma}.$$

* Neuberg, Sur les projections . . . d'un triangle fixe, Académie de Belgique, t. XLIV.
† Quarterly Journal, vol. 25 (1891), p. 186.

The difference will reduce to

$$i\sqrt{3}(e_\delta - e_\alpha) = \frac{\omega(\beta - \gamma)(\delta - \alpha)[A]}{B_\delta \cdot B_\alpha},$$

where

$$A = [(\delta\alpha + \beta\gamma)(\omega\gamma + \omega^2\beta) + \beta\gamma(\delta + \alpha)],$$

and

$$B_\delta = \gamma\alpha + \omega\alpha\beta + \omega^2\beta\gamma.$$

Similarly

$$i\sqrt{3}(e_\beta - e_\gamma) = \frac{\omega(\beta - \gamma)(\delta - \alpha)[A']}{B_\beta \cdot B_\gamma},$$

where

$$A' = [(\beta\gamma + \delta\alpha)(\omega^2\delta + \omega\alpha) + \delta\alpha(\beta + \gamma)].$$

The double ratio of the four equiangular points

$$\frac{(e_\delta - e_\alpha)(e_\beta - e_\gamma)}{(e_\alpha - e_\beta)(e_\gamma - e_\delta)},$$

will then have the form

$$\frac{(\beta - \gamma)^2(\delta - \alpha)^2[AA']}{B_\alpha \cdot B_\beta \cdot B_\gamma \cdot B_\delta} \times \frac{B_\alpha \cdot B_\beta \cdot B_\gamma \cdot B_\delta}{(\alpha - \beta)^2(\gamma - \delta)^2[CC']},$$

where C and C' are expressions similar in formation to A and A'. The factor

$$\frac{(\beta - \gamma)^2(\delta - \alpha)^2}{(\alpha - \beta)^2(\gamma - \delta)^2}$$

is real, since α, β, γ, δ are on a circle; moreover, since the conjugate of A is $\bar{A}$, the conjugate of the expression AA'/CC' is

$$\frac{\bar{A}\bar{A}'}{\bar{C}\bar{C}'} = \frac{\alpha^2\beta^2\gamma^2\delta^2 AA'}{\alpha^2\beta^2\gamma^2\delta^2 CC'};$$

thus the double ratio of the equiangular points is equal to its conjugate, and hence real. This will be true for the points e_i' as well as for e_i; therefore *each set of equiangular points is on a circle.*

7. **Incenters.** The triangle $\beta\gamma\delta$ will have four incenters, using the term in the general sense, of type I_α. The fact that the sixteen points I_α, I_β, I_γ, I_δ form a rectangular net has been discussed so recently, makes a bare reference sufficient.*

* The theorem was cited by Neuberg, in 1906; more recent proofs have been given by F. V. Morley, Amer. Math. Monthly, vol. 24, p. 430 (1917); N. Altshiller, Am. Math. Monthly, vol. 25, p. 412 (1918); and in the comprehensive article by J. W. Clawson, Annals of Math., vol. 20, p. 254 (1919). See also F. V. Morley, Amer. Math. Monthly, June 1920.

NOTE ON THE PRECEDING PAPER.

By F. Morley.

The interesting theorem of Section 6 in the preceding paper, that the Fermat or equiangular points of the 4 triangles formed from 4 concyclic points are arranged on circles, suggests the following treatment.

To obtain a convenient curve for handling metrically four points of a plane, a_1, a_2, a_3, a_4, we may look for points x such that the joins xa_i are apolar with the isotropic lines on x, taken twice. Analytically, if t_i be the clinant of xa_i, we are to have

$$s_2 = \Sigma\, t_1 t_2 = 0.$$

The locus of x is of the fourth order on the points a_i, with double points at the absolute points. Hence it is an elliptic quartic.

For an equiangular point, say e_4 of a_1, a_2, a_3, we have

$$t_1 + t_2 + t_3 = 0 = t_2 t_3 + t_3 t_1 + t_1 t_2,$$

so that

$$s_2 = 0.$$

That is, any four points and their eight equiangular points, are on a bicircular quartic.

Let u be an elliptic parameter on this curve. Then by Clebsch's statement of Abel's theorem, for four points on a circle

$$\begin{aligned} u_1 + u_2 + u_3 + u_4 &= \text{constant}, \\ &= 0, \text{ say}. \end{aligned}$$

Naming each point by its parameter and noting that the chord $a_1\, a_2$ subtends angles of $120°$ at e_3 and e_4, and therefore e_3, e_4, a_1, a_2, are on a circle, we have

$$a_1 + a_2 + e_3 + e_4 = 0$$

and similarly

$$a_3 + a_4 + e_1 + e_2 = 0.$$

If then the points a_i are on a circle,

$$a_1 + a_2 + a_3 + a_4 = 0,$$

so that

$$e_1 + e_2 + e_3 + e_4 = 0,$$

that is, the four equiangular points are on a circle.

In this way we have a configuration of circles arising from four concyclic points which is deserving of study.

QUALITATIVE PROPERTIES OF THE BALLISTIC TRAJECTORY.*

By T. H. Gronwall.

1. **Introduction.** In the following, we shall consider the projectile as a particle, and refer it to a right-handed system of rectangular coordinates with the origin at the muzzle of the gun, the positive x-axis being horizontal and directed toward the target and the positive y-axis vertical and directed upward. The retardation R due to the resistance of the air is introduced in its most general form as a function of the position and velocity of the projectile and the time, so that $R = R(x, y, z, x', y', z', t)$, where $x' = dx/dt$, etc. Denoting by $v = \sqrt{x'^2 + y'^2 + z'^2}$ the velocity of the projectile, the components of the retardation are Rx'/v, Ry'/v, Rz'/v, and g being the acceleration of gravity, the differential equations of motion are

$$x'' = -\frac{Rx'}{v}, \qquad y'' = -\frac{Ry'}{v} - g, \qquad z'' = -\frac{Rz'}{v},$$

with the initial conditions $x = y = z = 0$, $x' = x_0' > 0$, $y' = y_0' > 0$, $z' = 0$ for $t = 0$. The first and third equations give $(z'/x')' = 0$, so that $z' = \text{const.}\ x'$, and since $z' = z = 0$ for $t = 0$, it follows that $z = 0$ for any t, that is, the trajectory lies in the xy-plane.

Consequently the velocity is given by

$$v = \sqrt{x'^2 + y'^2},$$

and introducing the notation

$$E = E(x, y, x', y', t) = \frac{R}{v},$$

the differential equations of the trajectory become

$$(1) \qquad x'' = -Ex', \qquad y'' = -Ey' - g,$$

with the initial conditions for $t = 0$

$$(2) \qquad x = y = 0, \qquad x' = x_0' > 0, \qquad y' = y_0' > 0.$$

For purposes of comparison, let us consider briefly the trajectory in vacuum, where $E = 0$. Equations (1) and (2) then give

$$(3) \qquad x' = x_0', \qquad y' = y_0' - gt, \qquad x = x_0't, \qquad y = y_0't - \tfrac{1}{2}gt^2.$$

Besides the origin, two points of the trajectory are of special interest:

* Read before the American Mathematical Society, April 26, 1919 and February 28, 1920.

the summit, or point of maximum ordinate, and the point of fall, where $y = 0$. The coördinates, velocity components and time will be denoted by the subscript s at the summit and by the subscript ω at the point of fall.

By τ we denote the angle of slope defined by $\tan \tau = dy/dx$ and $-\pi/2 < \tau < \pi/2$, so that $x' = v \cos \tau$, $y' = v \sin \tau$; the value of τ for $t = 0$ is called the *angle of departure* and denoted by α, and for $t = t_\omega$, we write $\tau_\omega = -\omega$ and call ω the *angle of fall.* The abscissa x_ω of the point of fall is the *range* of the trajectory, and the corresponding t_ω is the *time of flight.* Eliminating t in (3), we obtain

$$y = \tan \alpha \cdot x - \frac{gx^2}{2x_0'^2}, \tag{4}$$

and since the summit and the point of fall are defined analytically by $y_s' = 0$ and $y_\omega = 0$ respectively, we have the following well-known formulas

$$\begin{aligned}
x_\omega &= \frac{2x_0'y_0'}{g} = \frac{v_0^2 \sin 2\alpha}{g}, \\
t_\omega &= \frac{2y_0'}{g} = \sqrt{\frac{2x_\omega \tan \alpha}{g}}, \\
\omega &= \alpha, \qquad x_\omega' = x_0', \qquad y_\omega' = -y_0', \\
t_s &= \frac{y_0'}{g} = \tfrac{1}{2}t_\omega, \\
y_s &= \frac{1}{2}\frac{y_0'^2}{g} = \tfrac{1}{2}gt_s^2 = \tfrac{1}{8}gt_\omega^2 = \tfrac{1}{4}x_\omega \tan \alpha, \\
\tau_s &= 0, \qquad x_s' = x_0', \qquad y_s' = 0.
\end{aligned} \tag{5}$$

Returning to the case of a resistance different from zero, we shall now derive qualitative properties of the trajectory under various hypotheses on the resistance. In § 2, hypothesis A, stating essentially that the resistance is positive, will be used to obtain a number of inequalities which all reduce to equalities of the type of (5) when the resistance is made equal to zero.* In § 3, the stronger hypothesis B, which states that the resistance depends on v and y alone, decreases or is stationary when y increases, and increases faster than v when the latter increases, is used to obtain additional inequalities of a similar character.

* Under the special assumption that the resistance is positive and depends on v alone, so that $E = E(v) > 0$, some of these properties have been obtained before. Thus properties I, III, VI VII, X, the first inequality in XI, XII, XIV and XV were given by P. de Saint-Robert, Mem. Ac. Sc. Torino, ser. 2, vol. 16 (1855). Properties XIII and the part of XVI stating that the time of flight is less on the rising than on the falling branch, were obtained by Zaboudsky in his Exterior Ballistics (St. Petersburg, 1895).

Finally, the still stronger hypothesis C, to the effect that the resistance is the product of a function of v by a function of y with some further restrictions on the mode of increase of these functions, is introduced in § 4 for the purpose of investigating the existence of maxima and minima of the velocity v.*

2. **Positive resistance—Hypothesis A.** All inequalities in this paragraph will be derived from

Hypothesis A: *For $x \geqq 0$, $x' > 0$, $t \geqq 0$ and all finite values of y and y', the function $E = E(x, y, x', y', t)$ is positive, and the derivatives $\partial E/\partial x$, $\partial E/\partial y$, $\partial E/\partial x'$ and $\partial E/\partial y'$ exist. Moreover, for any positive values of a, b, c and d, there exists an $M = M(a, b, c, d)$ such that the four derivatives and E itself are less in absolute value than M for $0 \leqq x \leqq a$, $0 < x' \leqq b$, $-c \leqq y \leqq c$, $-d \leqq y' \leqq d$ and $t \geqq 0$ but unrestricted upward.*

Since $x' > 0$ by equation (6) below, it follows from the general existence theorem for the solution of a system of differential equations, that the system (1) with the initial conditions (2) has then a unique solution which is defined (and finite) for any finite positive value of t.†

From the first of equations (1), we obtain

$$x' = x_0' e^{-\int_0^t E dt},$$

and since $E > 0$, it is seen at once that

I. *The horizontal velocity component x' is positive and decreases steadily from the value x_0' as t increases from zero.*

Since

$$x = \int_0^t x' dt \tag{7}$$

* In the special case $E = E(v)$, the result that v has at most one minimum but no maximum was obtained by de Saint-Robert (l.c.).

† When E is not bounded, the trajectory may end in a point which is reached in a finite time, beyond which x and y are no longer real. To show this, let us transform (1) to the independent variable τ; we then obtain the following equations, which are derived in all textbooks on ballistics:

$$\frac{d(v \cos \tau)}{d\tau} = -\frac{v^2 E}{g}, \quad \frac{dt}{d\tau} = -\frac{v}{g \cos \tau},$$
$$\frac{dx}{d\tau} = -\frac{v^2}{g}, \quad \frac{dy}{d\tau} = -\frac{v^2 \tan \tau}{g}.$$

Now make $E = gv^{-3}$, which becomes infinite for $x' = y' = 0$; then the first equation is integrable immediately and gives

$$(v \cos \tau)^2 = (v_0 \cos \alpha)^2 - 2 \sin \alpha + 2 \sin \tau.$$

Assume $(v_0 \cos \alpha)^2 < 2 \sin \alpha$ and determine τ_1 by

$$\sin \tau_1 = \sin \alpha - \tfrac{1}{2}(v_0 \cos \alpha)^2;$$

then $v \cos \tau$ is imaginary for $\tau < \tau_1$, and the motion ends at the time t_1 determined by

$$t_1 = \frac{1}{g}\int_{\tau_1}^{\alpha} \sqrt{2(\sin \tau - \sin \tau_1)} \frac{d\tau}{\cos^2 \tau}.$$

and $x' > 0$ by I, it follows that

II. *The horizontal distance x increases steadily from zero as t increases from zero.*

Eliminating E between the two equations (1), we find

$$\frac{d}{dt}\left(\frac{y'}{x'}\right) = -\frac{g}{x'}, \tag{8}$$

and integrating,

$$\tan\tau - \tan\alpha = \frac{y'}{x'} - \frac{y_0'}{x_0'} = -g\int_0^t \frac{dt}{x'} = -g\int_0^x \frac{dx}{x'^2}, \tag{9}$$

whence, since $x' > 0$,

III. *The slope $\tan\tau$ decreases as t increases (or as x increases) so that the trajectory is concave downward.*

We may replace (9) by

$$\frac{dy}{dx} = \tan\alpha - g\int_0^x \frac{dx}{x'^2}; \tag{10}$$

since $x' < x_0'$ by I, we have,

$$\frac{dy}{dx} < \tan\alpha - g\int_0^x \frac{dx}{x_0'^2} = \tan\alpha - g\frac{x}{x_0'^2}$$

and integrating between the limits 0 and x,

$$y < \tan\alpha\cdot x - g\frac{x^2}{2x_0'^2}, \tag{11}$$

whence by comparison to (4),

IV. *The trajectory lies below the trajectory in vacuum corresponding to the same initial velocity components x_0' and y_0'.*

At an extreme of y, we have $y' = 0$, and consequently $y'' = -g < 0$ by (1), so that every extreme is a maximum. Therefore y, being less, by IV, than the maximum ordinate $y_0'^2/2g$ of the trajectory in vacuum, has a unique maximum and no minimum, so that

V. *The altitude y increases from $t = 0$ to its maximum y_s at $t = t_s$, and decreases steadily as t increases beyond t_s.**

From the second of equations (1), it is seen that when $y' \geqq 0$, then $y'' < 0$ so that y' decreases as t increases, and since x' also decreases, the same is the case with $v = \sqrt{x'^2 + y'^2}$, or

VI. *The velocity and its vertical component decrease as t increases from 0 to t_s.*

* In case $y_0' < 0$, it follows in the same way that y decreases steadily as t increases from zero.

Since x' decreases, (9) gives

$$\frac{y'}{x'} < \frac{y_0'}{x_0'} - g\int_0^t \frac{dt}{x_0'} = \frac{y_0' - gt}{x_0'},$$

and consequently

$$\lim_{t\to\infty} \frac{y'}{x'} = -\infty \tag{12}$$

or

VII. *The angle of slope τ decreases steadily toward $-(\pi/2)$ as t increases indefinitely.*

We shall now prove

VIII. *The trajectory intersects the x-axis for one and only one $t = t_\omega$ beyond $t = t_s$, (so that the range x_ω exists).*

There can be only one point of intersection, the trajectory being concave downward (III), and since $y > 0$ for $t = t_s$, this point of intersection exists if we can show that $y \to -\infty$ as $t \to \infty$. By I, x' decreases toward a limit $\geqq 0$ as $t \to \infty$, which leads us to distinguish three cases. First, assume that $\lim\limits_{t\to\infty} x' > 0$; then (12) shows that there exists a t_1 such that $y' < -1$ for $t > t_1$ and consequently $y - y_1 < -(t - t_1)$, or $y \to -\infty$ as $t \to \infty$.* Second, assume that $\lim\limits_{t\to\infty} x' = 0$, but that there exists an $\epsilon > 0$ and a t_1 such that $y' < -\epsilon$ when $t > t_1$; then $y - y_1 < -\epsilon(t - t_1)$ and $y \to -\infty$ as before.† Finally, assume that $x' \to 0$ and y is bounded downward as $t \to \infty$, but that y' approaches zero infinitely often, that is, for any $\epsilon > 0$ however small, the set of t-values defined by $-\epsilon \leqq y' < 0$ contains values greater than an arbitrarily chosen t_1. Since y is bounded, it follows from (11) that x is also bounded, and since $0 < x' < x_0'$, hypothesis A shows that there exists an M such that $|E| < M$ for all values of t for which $-1 \leqq y' < 0$. Now take for ϵ the smaller of the values 1 and $g/2M$; from the second equation (1), we obtain

$$y'' \leqq |Ey'| - g < M\cdot\frac{g}{2M} - g = -\tfrac{1}{2}g$$

for all t such that $-\epsilon \leqq y' < 0$. But y' obviously cannot decrease for all t in this set; for some point belonging to this set, we must therefore

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* Example: $E = x' - a$, where $0 < a < x_0'$. The first equation (1) gives at once

$$\frac{x' - a}{x'} = \frac{x_0' - a}{x_0'}e^{-at},$$

whence $x' \to a$ as $t \to \infty$.

† Example: $E = 1$. Equations (1) give $x' = x_0'e^{-t}$, $y' + g = (y_0' + g)e^{-t}$, so that $x' \to 0$ and $y' \to -g$ as $t \to \infty$.

have $y'' \geqq 0$, in contradiction to $y'' < -\frac{1}{2}g$. Hence the last case cannot occur, and our theorem is proved.*

In order to derive further inequalities, we shall use the following theorem which is due to Tchebychef:

When $u(x)$ and $v(x)$ are continuous for $a \leqq x \leqq b$, and both these functions increase (or both decrease) as x increases, then

$$(b-a)\int_a^b u(x)v(x)dx > \int_a^b u(x)dx \cdot \int_a^b v(x)dx,$$

but

$$(b-a)\int_a^b u(x)v(x)dx < \int_a^b u(x)dx \cdot \int_a^b v(x)dx$$

when one of the functions increases and the other decreases.†

Returning to our trajectory, we form the expression

$$\frac{d}{dx}(y - x\tan\tau) = \frac{1}{x'}\frac{d}{dt}\left(y - x\frac{y'}{x'}\right) = -\frac{x}{x'}\frac{d}{dt}\left(\frac{y'}{x'}\right) = g\frac{x}{x'^2},$$

* That some assumption such as Hypothesis A is necessary to prove VIII is seen by the example $E = 3(1+t)^2$, where E increases indefinitely with t.

From equations (1),

$$x' = x_0' e^{1-(1+t)^3},$$

$$y' = e^{-(1+t)^3}\left[ey_0' - g\int_0^t e^{(1+u)^3}du\right].$$

When $0 < u < t$, we have $(1+u)^3 < (1+t)^2(1+u)$ and consequently

$$\int_0^t e^{(1+u)^3}du < \int_0^t e^{(1+t)^2(1+u)}du < \frac{e^{(1+t)^3}}{(1+t)^2},$$

so that

$$y' > y_0' e^{1-(1+t)^3} - \frac{g}{(1+t)^2},$$

$$y > y_0'\int_0^t e^{1-(1+t)^3}dt - g\left(1 - \frac{1}{1+t}\right).$$

For a sufficiently large y_0', it follows that $y > 0$ for large values of t, and consequently the trajectory does not intersect the x-axis (except at the origin).

† The following very simple proof was given by F. Franklin in the American Journal of Mathematics, vol. 7, p. 377 (1884): Consider the double integral

$$\int_a^b\int_a^b [u(t)-u(x)][v(t)-v(x)]dtdx = \int_a^b dt\int_a^b [u(t)v(t) - u(t)v(x) - u(x)v(t) + u(x)v(x)]dx$$

$$= (b-a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt\cdot\int_a^b v(x)dx - \int_a^b v(t)dt\cdot\int_a^b u(x)dx + (b-a)\int_a^b u(x)v(x)dx,$$

or replacing t by x in the single integrals to the right

$$\frac{1}{2}\int_a^b\int_a^b [u(t)-u(x)][v(t)-v(x)]dtdx = (b-a)\int_a^b u(x)v(x)dx - \int_a^b u(x)dx\cdot\int_a^b v(x)dx.$$

When $u(x)$ and $v(x)$ vary in the same sense, $u(t)-u(x)$ and $v(t)-v(x)$ have the same sign, so that their product is positive, and therefore also the double integral, which proves the first part of the theorem. The second part follows from the first upon replacing $u(x)$ by $-u(x)$.

the last step being a consequence of (8). Integrating from zero to x, we find

$$y = x \tan \tau + g \int_0^x \frac{x\,dx}{x'^2}. \tag{13}$$

In the integral, x increases and so does $1/x'^2$ by I; we may therefore apply Tchebychef's theorem and obtain

$$\begin{aligned} y &> x \tan \tau + g \cdot \frac{1}{x} \cdot \int_0^x x\,dx \cdot \int_0^x \frac{dx}{x'^2} \\ &= x \tan \tau + g \cdot \frac{1}{x} \cdot \frac{x^2}{2} \cdot - \frac{1}{g}(\tan \tau - \tan \alpha) \end{aligned}$$

by (9), or reducing,

$$\frac{y}{x} > \tfrac{1}{2}(\tan \alpha + \tan \tau).$$

Now move the origin to the point x_1, y_1 on the trajectory; x, y and α are then replaced by $x - x_1$, $y - y_1$ and τ_1, and our inequality becomes

$$\frac{y - y_1}{x - x_1} > \tfrac{1}{2}(\tan \tau_1 + \tan \tau) \tag{14}$$

or

IX. *The arithmetic mean of the slopes at any two points on the trajectory is less than the slope of the chord joining these two points.**

In the remainder of this paragraph, we shall denote by x_1, y_1 a point on the rising branch of the trajectory (where $y' > 0$), and by x_2, y_2 a point on the falling branch (where $y' < 0$), the corresponding times and velocity components being t_1, x_1', y_1' and t_2, x_2', y_2' respectively.

* This is also readily proved by the trapezoid formula with remainder term for the evaluation of definite integrals:

$$\int_{x_1}^{x_2} f(x)dx = \frac{x_2 - x_1}{2}[f(x_1) + f(x_2)] - \frac{1}{2}\int_{x_1}^{x_2}(x - x_1)(x_2 - x)f''(x)dx$$

which is verified at once by integrating by parts in the last integral. Since $x - x_1$ and $x_2 - x$ are positive in the interval of integration, it follows that when $f''(x) < 0$ for $x_1 < x < x_2$, then

$$\int_{x_1}^{x_2} f(x)dx > \frac{x_2 - x_1}{2}[f(x_1) + f(x_2)].$$

Now let $f(x) = \tan \tau$; then $f'(x) = -g/x'^2$ by (8), and

$$f''(x) = \frac{1}{x'}\frac{d}{dt}(-g/x'^2) = 2gx''/x'^4 = -2gE/x'^3$$

by the first of (1). Since $E > 0$, $x' > 0$, we have $f''(x) < 0$ and the last inequality becomes

$$y_2 - y_1 = \int_{x_1}^{x_2} \tan \tau\, dx > \frac{x_2 - x_1}{2}[\tan \tau_1 + \tan \tau_2],$$

which is identical with (14).

Making $x = x_2$, $y = y_2 = y_1$ in (14), we obtain $\tan \tau_1 + \tan \tau_2 < 0$, or

$$-\tau_2 > \tau_1, \tag{15}$$

and in particular, for $y_1 = y_2 = 0$,

$$\omega > \alpha, \tag{16}$$

whence

X. *At two points of equal altitude the slope of the trajectory is numerically greater on the falling than on the rising branch; in particular, the angle of fall ω is greater than the angle of departure α.*

Next, make $x_1 = y_1 = 0$, $x = x_s$ and $y = y_s$ in (14), whence $y_s/x_s > \frac{1}{2}\tan\alpha$, and furthermore $x_1 = x_s$, $y_1 = y_s$, $x = x_\omega$ and $y = 0$, whence $-y_s/(x_\omega - x_s) > \frac{1}{2}\tan\omega$, so that, combining these two inequalities,

$$\tfrac{1}{2}x_s \tan\alpha < y_s < \tfrac{1}{2}(x_\omega - x_s)\tan\omega. \tag{17}$$

Since $\tan\tau = dy/dx$, we have

$$x_1 = \int_0^{y_1} \frac{dy_1}{\tan\tau_1}, \qquad x_\omega - x_2 = \int_0^{y_2} \frac{dy_2}{-\tan\tau_2},$$

$$x_s - x_1 = \int_{y_1}^{y_s} \frac{dy_1}{\tan\tau_1}, \qquad x_2 - x_s = \int_{y_2}^{y_s} \frac{dy_2}{-\tan\tau_2},$$

and for $y_1 = y_2$, by means of (15),

$$x_1 > x_\omega - x_2, \qquad x_s - x_1 \geqq x_2 - x_s, \tag{18}$$

where the equality sign holds only when $x_1 = x_s = x_2$. In particular, for $x_1 = x_s = x_2$, (18) gives $x_s > \frac{1}{2}x_\omega$, and combining this with the outer terms in (17), we find

XI. *The abscissa x_s of the summit of the trajectory satisfies the inequalities*

$$\tfrac{1}{2}x_\omega < x_s < \frac{\tan\omega}{\tan\alpha + \tan\omega}\cdot x_\omega. \tag{19}$$

Introducing $x_s > \frac{1}{2}x_\omega$ in (17), we find

$$\tfrac{1}{4}x \tan\alpha_\omega < y_s < \tfrac{1}{4}x_\omega \tan\omega, \tag{20}$$

or

XII. *The maximum ordinate y_s lies between the maximum ordinates of the two trajectories in vacuum having the same range x_ω and the angles of departure α and ω respectively.**

* For another proof based on Rolle's theorem, see Charbonnier, Balistique extérieure rationnelle, vol. 1 (Paris, Doin, 1907), p. 193.

Upon multiplication by $2y'$, the second of equations (1) may be written $(y'^2)' = -2Ey'^2 - 2gy'$ or, since $dy/dt = y'$,

$$\frac{d}{dy} y'^2 = -2Ey' - 2g.$$

Integrating from y to y_s we find, since $y_s' = 0$,

$$y'^2 = \int_y^{y_s} 2Ey'dy + 2g(y_s - y).$$

Since y' is positive on the rising and negative on the falling branch, it follows that

$$y_1'^2 > 2g(y_s - y_1), \qquad y_2'^2 < 2g(y_s - y_2),$$

which may also be written, observing that y_2' is negative,

(21) $$y_1' > \sqrt{2g(y_s - y_1)}, \qquad -y_2' < \sqrt{2g(y_s - y_2)}.$$

When $y_1 = y_2$, (21) gives

(22) $$y_1' > -y_2'$$

or

XIII. *At two points of equal altitude on the trajectory, the vertical velocity component is numerically greater on the rising than on the falling branch.*

Since also $x_1' > x_2'$ by I, it follows from (22) that for $y_1 = y_2$

(23) $$v_1 > v_2,$$

or

XIV. *At two points of equal altitude on the trajectory, the velocity is greater on the rising than on the falling branch.*

For the arc s of the trajectory, we have $ds/dy = 1/\sin\tau$; hence, denoting by s_ω the total length of arc,

$$s_1 = \int_0^{y_1} \frac{dy_1}{\sin\tau_1}, \qquad s_\omega - s_2 = \int_0^{y_2} \frac{dy_2}{-\sin\tau_2},$$

and by (15), for $y_1 = y_2$,

(24) $$s_1 > s_\omega - s_2.$$

In particular, it follows from (24) that

XV. *The length of arc of the rising branch of the trajectory is greater than that of the falling branch.*

On account of $dt/dy = 1/y'$, we have

$$t_s - t_1 = \int_{y_1}^{y_s} \frac{dy_1}{y_1'}, \qquad t_2 - t_s = \int_{y_2}^{y_s} \frac{dy_2}{-y_2'},$$

and by (21)

$$t_s - t_1 < \int_{y_1}^{y_s} \frac{dy_1}{\sqrt{2g(y_s - y_1)}}, \qquad t_2 - t_s > \int_{y_2}^{y_s} \frac{dy_2}{\sqrt{2g(y_s - y_2)}},$$

or performing the integrations

(25) $$t_s - t_1 < \sqrt{\frac{2(y_s - y_1)}{g}}, \qquad t_2 - t_s > \sqrt{\frac{2(y_s - y_2)}{g}}.$$

In particular, for $y_1 = y_2 = 0$

(26) $$t_s < \sqrt{\frac{2y_s}{g}} < t_\omega - t_s,$$

or

XVI. *The time of flight on the rising branch is less, and that on the falling branch greater, than the corresponding time of flight on a trajectory in vacuum with the same maximum ordinate.*

It should be noted in this connection that the empirical inequality

$$y_s > \tfrac{1}{8} g t_\omega^2$$

proposed by several ballisticians, cannot be universally true. On the contrary, the inequality sign must be reversed when the angle of departure is sufficiently small, as will be shown on another occasion, in connection with the use of power series in exterior ballistics.

To obtain inequalities connecting t_ω with x_ω, we write

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} x' \frac{dt}{x'}$$

and apply Tchebychef's theorem:

$$t_2 - t_1 < \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x' dt \cdot \int_{t_1}^{t_2} \frac{dt}{x'} = \frac{1}{t_2 - t_1} (x_2 - x_1) \cdot \frac{1}{g} (\tan \tau_1 - \tan \tau_2)$$

whence

(27) $$t_2 - t_1 < \sqrt{\frac{x_2 - x_1}{g} (\tan \tau_1 - \tan \tau_2)}.*$$

Applying (27) to the rising and falling branches of the trajectory, we find

$$t_s < \sqrt{\frac{x_s \tan \alpha}{g}}, \qquad t_\omega - t_s < \sqrt{\frac{(x_\omega - x_s) \tan \omega}{g}},$$

* Another proof of (27) is obtained from

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{x'}$$

by the application of Schwarz' inequality:

$$(t_2 - t_1)^2 < \int_{x_1}^{x_2} dx \, . \int_{x_1}^{x_2} \frac{dx}{x'^2} = (x_2 - x_1) \, . \, \frac{1}{g} (\tan \tau_1 - \tan \tau_2).$$

and by addition

$$t_\omega < \sqrt{\frac{x_s \tan\alpha}{g}} + \sqrt{\frac{(x_\omega - x_s)\tan\omega}{g}}. \tag{28}$$

Replacing x_s by $\frac{1}{2}x_\omega$ increases the expression to the right; in fact, it follows from $x_s > \frac{1}{2}x_\omega$ and $\tan\alpha < \tan\omega$ that

$$\begin{aligned}\sqrt{x_s \tan\alpha} + \sqrt{(x_\omega - x_s)\tan\omega} - (\sqrt{\tfrac{1}{2}x_\omega \tan\alpha} + \sqrt{\tfrac{1}{2}x_\omega \tan\omega})\\ = (\sqrt{x_s} - \sqrt{\tfrac{1}{2}x_\omega})\sqrt{\tan\alpha} - (\sqrt{\tfrac{1}{2}x_\omega} - \sqrt{x_\omega - x_s})\sqrt{\tan\omega}\\ < \sqrt{\tan\alpha}(\sqrt{x_s} + \sqrt{x_\omega - x_s} - 2\sqrt{\tfrac{1}{2}x_\omega}),\end{aligned}$$

and the last factor is negative by the algebraic identity

$$(\sqrt{x_s} + \sqrt{x_\omega - x_s} + \sqrt{2x_\omega})(\sqrt{x_s} + \sqrt{x_\omega - x_s} - \sqrt{2x_\omega}) = -(\sqrt{x_s} - \sqrt{x_\omega - x_s})^2.$$

Therefore (28) may be replaced by

$$t_\omega < \sqrt{\frac{x_\omega}{2g}}(\sqrt{\tan\alpha} + \sqrt{\tan\omega}).$$

On the other hand, we have since x' decreases,

$$\tan\alpha = g\int_0^{t_s}\frac{dt}{x'} < g\frac{t_s}{x_s'},$$

$$x_\omega - x_s = \int_{t_s}^{t_\omega} x'dt < x_s'(t_\omega - t_s),$$

and multiplying these two inequalities

$$t_s(t_\omega - t_s) > \frac{x_\omega - x_s}{g}\tan\alpha.$$

But $t_s(t_\omega - t_s) < \frac{1}{4}t_\omega^2$, and $x_\omega - x_s > x_\omega \tan\alpha\,(\tan\alpha + \tan\omega)$ by (19), so that

$$\tfrac{1}{4}t_\omega^2 > \frac{x_\omega \tan^2\alpha}{g(\tan\alpha + \tan\omega)},$$

and consequently we find that

XVII. *The time of flight is bounded in terms of the range by the inequalities*

$$\sqrt{\frac{x_\omega}{2g}}\cdot\sqrt{\frac{8\tan^2\alpha}{\tan\alpha + \tan\omega}} < t_\omega < \sqrt{\frac{x_\omega}{2g}}(\sqrt{\tan\alpha} + \sqrt{\tan\omega}). \tag{29}$$

3. **Resistance depending on v and y only. Hypothesis B.** All properties in this paragraph will be derived from the following hypothesis (except XIX, which requires the stronger assumption stated):

HYPOTHESIS B: *The function $E = E(v, y)$ depends on v and y alone. For $v > 0$ and all finite values of y it is positive and has derivatives $\partial E/\partial v$ and $\partial E/\partial y$, and for any positive values of a and c, there exists an $M = M(a, c)$ such that the two derivatives and E itself are less in absolute value than M for $0 < v \leqq a$, $-c \leqq y \leqq c$. Moreover,*

$$\frac{\partial E}{\partial v} > 0, \qquad \frac{\partial E}{\partial y} \leqq 0,$$

so that E increases when v increases, but does not increase when y increases.

We shall first derive some inequalities involving the horizontal velocity components at the origin, the summit and the point of fall. From the first of equations (1), we obtain

$$x_0' - x_s' = \int_0^{t_s} Ex'dt = \int_0^{y_s} \frac{E(v_1, y_1)}{\tan \tau_1} dy_1,$$

and similarly

$$x_s' - x_\omega' = \int_0^{y_s} \frac{E(v_2, y_2)}{-\tan \tau_2} dy_2.$$

For $y_1 = y_2$, we have $\tan \tau_1 < -\tan \tau_2$ by X, and $v_1 > v_2$ by XIV, so that $E(v_1, y_1) > E(v_2, y_2)$ by hypothesis B, and consequently $x_0' - x_s' > x_s' - x_\omega'$ or

XVIII. *The horizontal velocity component at the summit is less than the arithmetic mean of its values at the origin and at the point of fall.*

$$x_s' < \tfrac{1}{2}(x_0' + x_\omega'). \tag{30}$$

From the first of equations (1), it follows that

$$\log \frac{x_0'}{x_s'} = \int_0^{t_s} -\frac{x''}{x'} dt = \int_0^{t_s} Edt = \int_0^{y_s} \frac{E(v_1, y_1)}{y_1'} dy_1 = \int_0^{y_s} \frac{E(v_1, y_1)}{v_1} \frac{dy_1}{\sin \tau_1},$$

and similarly

$$\log \frac{x_s'}{x_\omega'} = \int_0^{y_s} \frac{E(v_2, y_2)}{v_2} \frac{dy_2}{-\sin \tau_2}.$$

If, for $y_1 = y_2$ (whence $v_1 > v_2$), we have $E(v_1, y_1)/v_1 \geqq E(v_2, y_2)/v_2$, it follows that $\log (x_0'/x_s') > \log (x_s'/x_\omega')$, or

XIX. *When, in addition to hypothesis* B, *$E(v, y)$ is such that $E(v, y)/v$ does not decrease when v increases and y remains constant, the horizontal velocity component at the summit is less than the geometric mean of its values at the origin and at the point of fall:*

$$x_s' < \sqrt{x_0' x_\omega'}. \tag{31}$$

In the equation

$$x_0' - x_s' = \int_0^{t_s} Ex'dt,$$

x' decreases, and E also decreases by hypothesis B, since v decreases (VI) and y increases; hence Tchebychef's theorem gives

$$x_0' - x_s' > \frac{1}{t_s} \cdot \int_0^{t_s} x'dt \cdot \int_0^{t_s} Edt = \frac{x_s}{t_s} \log\left(\frac{x_0'}{x_s'}\right),$$

or

$$x_s < t_s \frac{x_0' - x_s'}{\log x_0' - \log x_s'}. \tag{32}$$

Similarly the equation

$$\frac{1}{x_s'} - \frac{1}{x_0'} = \int_0^{t_s} -\frac{x''}{x'^2}dt = \int_0^{t_s} E\frac{dt}{x'}$$

gives

$$\frac{1}{x_s'} - \frac{1}{x_0'} < \frac{1}{t_s} \cdot \log\left(\frac{x_0'}{x_s'}\right) \cdot \frac{\tan\alpha}{g}$$

or

$$t_s < x_0'x_s' \frac{\log x_0' - \log x_s'}{x_0' - x_s'} \frac{\tan\alpha}{g} \tag{33}$$

whence by (32)

$$x_s < x_0'x_s' \frac{\tan\alpha}{g} \tag{34}$$

and since $x_s > \frac{1}{2}x_\omega$ by (19), we have

$$x_s' > \frac{gx_\omega \cot\alpha}{2x_0'}. \tag{35}$$

Further inequalities of this type may be obtained, but they are too complicated to be of much interest.

We now proceed to derive some properties of the trajectory when t increases indefinitely.

XX. *For all values of t, the velocity is bounded by*

$$v \leqq \max\left(v_0, \frac{g}{E(v_0, 0)}\right). \tag{36}$$

Since $vv' = x'x'' + y'y''$, we obtain from (1)

$$v' = -vE - g\frac{y'}{v}. \tag{37}$$

By VI and XIV, $v < v_0$ for $0 < t \leqq t_\omega$, and we need therefore only show that the upper bound (36) cannot be reached for $t > t_\omega$. Assume t_1 to be the smallest value of t beyond t_ω for which the upper bound (36) is

reached; since $v_\omega < v_0$, it follows that v must either increase through t_1 or have a maximum there, so that $v' \geqq 0$ at t_1. But at t_1, we have $v \geqq v_0$ and $y < 0$, hence $E(v, y) \geqq E(v_0, 0)$ by hypothesis B, and

$$- y'/v = - \sin \tau < 1$$

by VII, so that (37) gives $v' < g - vE(v_0, 0)$. Since we have assumed that $v \geqq g/E(v_0, 0)$ at t_1, it follows that $v' < 0$ for this value of t, and this contradiction to $v' \geqq 0$ proves our theorem.

Equation (8) may be written

$$\frac{d\tau}{dt} = - \frac{g}{x'} \cos^2 \tau = - g \frac{x'}{v^2},$$

and therefore $dx/d\tau = - v^2/g$ or

$$x = \frac{1}{g} \int_\tau^\alpha v^2 d\tau.$$

Now v^2 is bounded by XX, and $\tau > - \pi/2$; consequently x is bounded and being an increasing function of t, x tends towards the limit given by making $\tau = - \pi/2$ in the above integral. Hence

XXI. *The falling branch of the tràjectory has the vertical asymptote*

$$x = \frac{1}{g} \int_{-(\pi/2)}^\alpha v^2 d\tau.$$

Since $x = \int_0^t x' d\tau$, and x' is positive and decreasing, it follows from the boundedness of x that

$$x' \to 0 \text{ as } t \to \infty.$$

4. **Maxima and minima of the velocity. Hypothesis C.** It is the main purpose of this paragraph to show that, under fairly weak assumptions, there exist no extremes of v beyond a certain value of t. If there are infinitely many extremes, the points at which they occur must therefore have all their limiting points at finite distance, and by a stronger assumption, this possibility may also be excluded, so that v has then only a finite number of extremes.

The first step in this investigation consists in showing that $vE \to g$ as $t \to \infty$. Suppose first that there exist a t_1 such that for $t > t_1$, vE either increases or decreases steadily as t increases. Since $vE > 0$, it then follows that vE either increases indefinitely or tends toward a limit greater than or equal to zero. Now $- y'/v = \sin (- \tau) \to 1$ by VII, and unless $vE \to g$, it then follows from (37) that there exist an $\epsilon > 0$ and a t_2 such that either $v' > \epsilon$ for $t > t_2$, or $v' < - \epsilon$ for $t > t_2$, according as $\lim vE < g$ or $\lim vE > g$ (including $vE \to \infty$). Consequently we have either

$v > v_2 + \epsilon(t - t_2) \to \infty$ as $t \to \infty$, in contradiction to XX, or $v < v_2 - \epsilon(t - t_2) \to -\infty$ in contradiction to $v \geqq 0$. Hence $vE \to g$ under the assumption made.

There remains to be considered the case when extremes of vE occur for indefinitely increasing values of t. It is obvious that in this case, lim inf vE as $t \to \infty$ equals the inferior limit of the minima of vE, and lim sup vE as $t \to \infty$ equals the superior limit of the maxima. Now

$$\frac{d(vE)}{dt} = vE\left[\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)v' + \frac{1}{E}\frac{\partial E}{\partial y}y'\right]$$
$$= vE\left[-\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)\left(vE + g\frac{y'}{v}\right) + \frac{1}{E}\frac{\partial E}{\partial y}y'\right]$$

by (37), so that at an extreme of vE

$$\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)\left(vE + g\frac{y'}{v}\right) = -\frac{1}{E}\frac{\partial E}{\partial y}\cdot - y'.$$

Since $\partial E/\partial v > 0$, $\partial E/\partial y \leqq 0$ by hypothesis B, it follows that for $y' \geqq 0$, the expression to the left is positive and that to the right negative or zero, so that the extremes of vE occur for $y' < 0$. The first factor to the left and the right hand member being then both positive, it follows that $vE + gy'/v > 0$, and since $\partial E/\partial v > 0$, we have

$$\frac{1}{v}\left(vE + g\frac{y'}{v}\right) < -\frac{1}{E}\frac{\partial E}{\partial y}\cdot - y';$$

moreover, $0 < -y' < v$, so that finally

$$-g\frac{y'}{v} < vE < -g\frac{y'}{v} - \frac{1}{E}\frac{\partial E}{\partial y}\cdot v^2 \tag{38}$$

at an extreme of vE.

On account of $-y'/v \to 1$ as $t \to \infty$, it follows from (38) that under hypothesis B

$$\liminf_{t\to\infty} vE \geqq g. \tag{39}$$

Before proceeding further, we shall prove the following

Lemma. Let $y(x)$ and $z(x)$ be solutions of the differential equations

$$\frac{dy}{dx} = f(x, y), \qquad \frac{dz}{dx} = \varphi(x, z)$$

with the initial conditions $y = z = y_0$ for $x = x_0$, and assume that $f(x, u)$ and $\varphi(x, u)$ are continuous for $x \geqq x_0$ and $|u - y(x)| < \epsilon$ (or for $x \geqq x_0$ and $|u - z(x)| < \epsilon$), and that moreover $f(x, u) < \varphi(x, u)$ for $x \geqq x_0$ and

$u = y(x)$ (*or for* $x \geqq x_0$ *and* $u = z(x)$). *Then*

$$y(x) < z(x)$$

for $x > x_0$.

Since, by hypothesis, $f(x, u)$ and $\varphi(x, u)$ are continuous at $x = x_0$, $u = y_0$, and $f(x_0, y_0) - \varphi(x_0, y_0) < 0$, it follows that $f(x, y(x)) - \varphi(x, z(x)) < 0$ for $x_0 \leqq x \leqq x_0 + \delta$, when δ is sufficiently small, so that $dy/dx < dz/dx$ and consequently $y < z$ for $x_0 < x < x_0 + \delta$. Now assume that $y(x) < z(x)$ for $x_0 < x < x_1$, but $y(x_1) = z(x_1)$. Then the same argument as before shows that $y(x) < z(x)$ for $x_1 < x < x_1 + \delta$, where δ is sufficiently small, and writing $v(x) = y(x) - z(x)$, we have for h positive and sufficiently small, $v(x_1 - h) < 0$, $v(x_1) = 0$, $v(x_1 + h) < 0$, so that

$$\frac{v(x_1 - h) - v(x_1)}{-h} > 0 > \frac{v(x_1 + h) - v(x_1)}{h},$$

and for $h \to 0$, we find $dv/dx = 0$ for $x = x_1$. But we have

$$\left(\frac{dv}{dx}\right)_{x=x_1} = f(x_1, y(x_1)) - \varphi(x_1, z(x_1))$$
$$= f(x_1, y(x_1)) - \varphi(x_1, y(x_1)) < 0$$

by hypothesis, and this contradiction shows that there is no $x_1 > x_0$ for which $y(x_1) = z(x_1)$, so that $y(x) < z(x)$ for $x > x_0$.

As an application, write $\eta = -y$, so that $\eta \to \infty$ as $t \to \infty$, and compare the equation

$$\frac{dv}{d\eta} = \frac{g}{v} - \frac{v}{\eta'} E(v, -\eta) \tag{40}$$

obtained from (37) by a change of variable, with the equation

$$\frac{dV}{d\eta} = \frac{g}{V} - E(V, -\eta), \tag{41}$$

the initial conditions being $v = V = v_1$ for $\eta = \eta_1$ and η_1 belonging to a point on the falling branch of the trajectory. Since $0 < -y' = \eta' < v$, we have

$$\frac{g}{v} - \frac{v'}{\eta} E(v, -\eta) < \frac{g}{v} - E(v, -\eta),$$

and by the lemma, it follows that

$$v(\eta) < V(\eta) \qquad \text{for} \qquad \eta > \eta_1. \tag{42}$$

The mechanical interpretation of this is obvious, since (41) is the equation of motion of a particle projected vertically downward.

Now suppose that V has a minimum $= V_2$ for $\eta = \eta_2$; then, by (41),

$V_2E(V_2, -\eta_2) = g$. For $\eta_2 < \eta < \eta_2 + \delta$ and δ sufficiently small, we have $V > V_2$ since V_2 is a minimum, and by hypothesis B, it follows that $VE(V, -\eta) > V_2E(V_2, -\eta) \geqq V_2E(V_2, -\eta_2) = g$, so that, by (41), $dV/d\eta < 0$ or V decreases, which contradicts the assumption of a minimum. Hence V has at most one extreme, which is a maximum, and therefore, as $t \to \infty$ or $\eta \to \infty$, V tends either to a finite limit or to infinity, so that η sufficiently large, $V \geqq c$ where $c \geqq 0$. Now assume that $E(c, y) \to \infty$ as $y \to -\infty$ for any positive value of the constant c; then if $V \geqq c > 0$, it follows from (41) that $dV/d\eta \to -\infty$ as $y \to -\infty$ and consequently $V \to -\infty$ which is impossible. Consequently $V \to 0$ as $t \to \infty$ and a fortiori $v \to 0$ by (42), or

XXII. *When in addition to hypothesis B, we assume that $E(c, y) \to \infty$ as $y \to -\infty$ for any positive value however small of the constant c, then*

$$v \to 0 \text{ as } t \to \infty.$$

Returning to the proof of $vE \to g$, we now introduce

HYPOTHESIS C. *The function $E(v, y)$ has the form*

$$E(v, y) = G(v) \cdot H(y),$$

where $G(v) > 0$ and $G'(v)$ exists and is > 0 for $v > 0$, and is bounded for $0 < v < a$, where a is as large as we please; moreover $H(y) > 0$, $H(y) \to \infty$ as $y \to -\infty$, $H'(y)$ exists and is negative and bounded for $-b < y < b$, where b is as large as we please, and $h(y) = -d\,[\log H(y)]/dy$ satisfies the condition

$$e^{-cy}h(y) \to \infty \text{ as } y \to -\infty$$

for any positive constant c however small. Finally, any one of the three following assumptions is made:

1) *$h(y)$ is bounded as $y \to -\infty$; no additional condition on $G(v)$.*

2) $\dfrac{h(y)}{H(y)} \to 0$ *as $y \to -\infty$, and $G(0) = 0$.*

3) *For some constant $m \geqq 1$, and some positive constant c_1,*

$$\frac{h(y)}{[H(y)]^{2(m+1)}} \to 0 \text{ as } y \to -\infty,$$

and $G(v) > c_1v^m$ for v sufficiently small.

This form of $E(v, y)$ is the one used in actual computations, $G(v)$ being given empirically in form of a table and assumed to satisfy the conditions $C1$. The factor $H(y)$ is introduced to account for the decrease in air resistance due to the decrease of the density of the air with increasing altitude; the expression formerly used $H(y) = (1 - ky)^n$, where k and

n are positive constants, has been replaced recently by $H(y) = e^{-hy}$, where h is a positive constant. It is seen at once that both these expressions for $H(y)$ satisfy $C1$.

Under hypothesis $C1$, it follows from (38) and XXII that for the maxima of vE

$$vE < -g\frac{y'}{v} + h(y)\cdot v^2 \to g \text{ as } t \to \infty,$$

so that $\limsup vE \leqq g$ as $t \to \infty$, and together with (39) this gives $vE \to g$.

Under hypothesis C2, it is sufficient to prove $\limsup VE(V, -\eta) \leqq g$ as $\eta \to \infty$, since for the same value of $\eta > \eta_1$, $v < V$ by (42) and consequently $vE(v, -\eta) < VE(V, -\eta)$.

From (41) we obtain

$$\frac{d\,VG(V)}{d\eta} = \frac{G(V) + VG'(V)}{V}[g - VG(V)\,H(-\eta)]. \tag{43}$$

Since $G(0) = 0$, we have $G(V) = G(V) - G(0) = VG'(\theta V)$, $0 < \theta < 1$, and since G' is bounded (V being bounded, tending toward zero as $\eta \to \infty$), it follows that

$$0 < \frac{G(V) + VG'(V)}{V} < k,$$

where k is a constant. With the equation of comparison

$$\frac{d\xi}{d\eta} = k[g - \xi H(-\eta)] \tag{44}$$

and the initial conditions $\xi = VG(V)$ for $\eta = \eta_1$, it follows from the lemma that $VG(V) < \xi$ for $\eta > \eta_1$, so that all we need to prove is

$$\limsup_{\eta\to\infty} H(-\eta)\cdot\xi \leqq g. \tag{45}$$

The solution of (44) taking the value ξ_1 for $\eta = \eta_1$ is

$$\xi = e^{-\int_{\eta_1}^{\eta}kH\,d\eta}\left[\xi_1 + g\int_{\eta_1}^{\eta_2} ke^{\int_{\eta_1}^{\eta}kH\,d\eta}d\eta + g\int_{\eta_2}^{\eta} ke^{\int_{\eta_1}^{\eta}kH\,d\eta}d\eta\right],$$

where η_2 is any constant greater than η_1, and $H = H(-\eta)$. Integrating by parts in the last integral, we find

$$\int_{\eta_2}^{\eta} ke^{\int_{\eta_1}^{\eta}kH\,d\eta}d\eta = \int_{\eta_2}^{\eta}\frac{1}{H(-\eta)}\frac{d}{d\eta}[e^{\int_{\eta_1}^{\eta}kH\,d\eta}]d\eta$$
$$= \left[\frac{1}{H}e^{\int_{\eta_1}^{\eta}kH\,d\eta}\right]_{\eta_2}^{\eta} + \int_{\eta_2}^{\eta}\frac{h(-\eta)}{H(-\eta)}e^{\int_{\eta_1}^{\eta}kH\,d\eta}d\eta,$$

and consequently

$$
(46)\quad
\begin{aligned}
H(-\eta)\cdot\xi - g &= gHe^{-\int_{\eta_1}^{\eta} kHd\eta}\int_{\eta_2}^{\eta}\frac{h}{H}e^{\int_{\eta_1}^{\eta} kHd\eta}d\eta \\
&+ He^{-\int_{\eta_1}^{\eta} kHd\eta}\left[\xi_1 + g\int_{\eta_1}^{\eta_2} ke^{\int_{\eta_1}^{\eta} kHd\eta}d\eta - \frac{g}{H(-\eta_2)}e^{\int_{\eta_1}^{\eta_2} kHd\eta}\right].
\end{aligned}
$$

For $\eta > \eta_2$, the first term to the right is less than

$$
\frac{g\displaystyle\int_{\eta_2}^{\eta}\frac{h}{H}e^{\int_{\eta_1}^{\eta} kHd\eta}d\eta}{\dfrac{1}{H(-\eta)}e^{\int_{\eta_1}^{\eta} kHd\eta} - \dfrac{1}{H(-\eta_2)}e^{\int_{\eta_1}^{\eta_2} kHd\eta}},
$$

and by the well-known formula

$$
\frac{\varphi(\eta) - \varphi(\eta_2)}{\psi(\eta) - \psi(\eta_2)} = \frac{\varphi'(\eta_3)}{\psi'(\eta_3)},
$$

where η_3 is some value between η_2 and η, this expression equals

$$
\frac{g\dfrac{h(-\eta_3)}{H(-\eta_3)}}{k - \dfrac{h(-\eta_3)}{H(-\eta_3)}},
$$

and since $\eta_3 > \eta_2$, it follows from hypothesis C2 that this expression may be made less than any $\epsilon > 0$ however small by taking η_2 sufficiently large. Having thus fixed η_2, the second term to the right in (46) approaches zero as $\eta \to \infty$, since

$$
He^{-\int_{\eta_1}^{\eta} kHd\eta} \to 0 \text{ as } \eta \to \infty,
$$

the logarithmic derivative of this expression being

$$
H(-\eta)\left[\frac{h(-\eta)}{H(-\eta)} - k\right],
$$

which approaches $-\infty$ when $\eta \to \infty$ by hypothesis C2. Consequently, (46) gives

$$
\limsup H(-\eta)\cdot\xi - g \leqq \epsilon,
$$

which is equivalent to (45), and hence $vE \to g$ as $t \to \infty$.

Under hypothesis C3, we have

$$
\frac{g}{V} - G(V)\,H(-\eta) < \frac{g}{V} - c_1 V^m H(-\eta),
$$

and comparing V defined by (41), to W defined by

$$\frac{dW}{d\eta} = \frac{g}{W} - c_1 W^m H(-\eta) \tag{47}$$

and the initial condition $W = V = v_1$ for $\eta = \eta_1$, we have $v < V < W$ for $\eta > \eta_1$. But in (47), we have $G(W) = c_1 W^m$ so that $G(0) = 0$, and hypothesis C3 is stronger in respect to H than C2; hence we have, from what has been proved under hypothesis C2,

$$\limsup_{\eta \to \infty} c_1 W^{m+1} H(-\eta) \leqq g,$$

and consequently, for η sufficiently large

$$c_1 W^{m+1} H(-\eta) < 2g.$$

Therefore

$$v^2 h(y) < V^2 h(y) < W^2 h(y) < \left(\frac{2g}{c_1}\right)^{2/(m+1)} \cdot \frac{h(y)}{(H(y))^{2/(m+1)}} \to 0$$

as $y \to -\infty$ by C3, and from (38) we conclude that the upper limit of the maxima of vE does not exceed g. Hence $vE \to g$ under hypothesis C3, so that we have the theorem:

XXIII. *Under hypothesis C, the retardation tends toward the limit g on the falling branch of the trajectory*:

$$vE(v, y) \to g \text{ as } t \to \infty. \tag{48}$$

At an extreme of v, we have $v' = 0$, or

$$vE = -g\frac{y'}{v} \tag{49}$$

by (37), and

$$\begin{aligned} v'' &= -\frac{\partial}{\partial v}\left(vE + g\frac{y'}{v}\right) \cdot v' - vE\frac{\partial \log E}{\partial y} y' - g\frac{y''}{v} \\ &= -\frac{\partial}{\partial v}\left(vE + g\frac{y'}{v}\right) \cdot v' + vEh(y) \cdot y' + \frac{g}{v}(Ey' + g); \end{aligned}$$

eliminating vE by (49), we find that for $v' = 0$,

$$v'' = \frac{g}{v^3}(gx'^2 - v^2 y'^2 h(y)). \tag{50}$$

For a minimum of v, we have $v'' \geqq 0$, and since $h(y) > 0$ by hypothesis C and $v^2 > y'^2$, it follows from (50) that

$$h(y)\left(\frac{y'^2}{x'}\right)^2 < g. \tag{51}$$

From equations (1), it is readily seen that

$$\frac{d}{dt}\log\frac{y'^2}{x'} = -\frac{1}{y'}(2g + Ey'),$$

or, since $\eta = -y$,

$$\frac{d}{d\eta}\log\frac{y'^2}{x'} = \frac{1}{y'^2}(2g + Ey').$$

Since $Ey' \to -Ev \to -g$ by (48), and $y'^2 < v^2$ is bounded by XX, it follows that there exists a positive constant c such that, for η_1 sufficiently large,

$$\frac{d}{d\eta}\log\frac{y'^2}{x'} > \frac{c}{2} \qquad \text{for} \qquad \eta \geqq \eta_1,$$

and consequently

$$\frac{y'^2}{x'} > \left(\frac{y'^2}{x'}\right)_1 e^{c(\eta-\eta_1)/2}.$$

Since $e^{c\eta}h(-\eta) \to \infty$ as $\eta \to \infty$ by hypothesis C, it follows that

$$h(y)\left(\frac{y'^2}{x'}\right)^2 \to \infty \qquad \text{as} \qquad t \to \infty,$$

and comparing this to (51), it is seen that for sufficienlty large values of t, no minima of v can occur, and hence no maxima, since these alternate with the minima. Therefore

XXIV. *Under hypothesis C, there exists a t_1 such that for $t > t_1$, the velocity v decreases steadily toward zero as t increases.**

If v has an infinity of extremes, the points at which they occur must consequently have a limiting point t_1 at finite distance. By Rolle's theorem, the zeros of v'' separate those of v', and consequently t_1 is a limiting point of zeros of v'', and similarly of as many of the higher derivatives as may exist. By continuity, we therefore have $v'(t_1) = 0$, $v''(t_1) = 0$, $v'''(t_1) = 0, \cdots$.

Assuming the second derivatives of $G(v)$ and $H(y)$ to exist, v''' exists. Differentiating (37) twice, substituting the values of x''' and y''' obtained from (1), making $v' = v'' = 0$, and eliminating E by $v' = 0$ or (49), it is shown without difficulty that

$$\tfrac{1}{2}(v^2)''' = -2h(y)g^2\frac{y'^3}{v^2} + h(y)gy' + h(y)^2gy'^3 - g^3\frac{y'^3}{v^4} + g^3\frac{y'}{v^2} - h'(y)g\frac{y'^3}{v^2}.$$

Equating this expression to zero, and making $v'' = 0$ in (50), we obtain

* Under the assumption (contrary to C) that $h(y) \leqq 0$, it follows from (50) that $v'' > 0$, that is, every extreme is a minimum. The velocity has therefore one minimum (since it decreases on the rising branch) and no maximum, or else it decreases steadily from $t = 0$ onward.

after some algebraic reductions

$$2h(y)^2 - \frac{h'(y)}{v^2} = 0.$$

When $h'(y) \leqq 0$, this equation is impossible, so that XXIV may be replaced by the following:

XXV. *When, in addition to hypothesis* C, *we have* $h'(y) \leqq 0$, *the velocity* v *has only a finite number of extremes.*

In particular, this is true under the assumption made in practice, $h(y) = h =$ a positive constant.

Since v decreases on the rising branch, the first extreme is necessarily a minimum, and from $v \to 0$ as $t \to \infty$ it follows that the last extreme must be a maximum. In many computed trajectories, a minimum followed by a maximum occurs for $y > 0$, or before the projectile strikes the ground (the part of the falling branch where $y < 0$ being of course of no practical interest). In no case, however, has a second minimum been discovered, and it is therefore plausible enough that v has only one minimum and one maximum, even for $y < 0$. The question of proving or disproving this conjecture remains open.

TECHNICAL STAFF,
OFFICE OF THE CHIEF OF ORDNANCE.

THE MEAN OF A FUNCTIONAL OF ARBITRARY ELEMENTS.

BY NORBERT WIENER.

1. P. J. Daniell* has recently developed a powerful method whereby if the notion of integration is once defined for a very restricted set of functions of arbitrary elements, it can be extended to a much more comprehensive set of functions. Daniell himself in his second article applied his method to the discussion of integrals in a denumerable infinity of dimensions. Daniell's method, however, leaves the mode of establishing integration over the original restricted set in general undetermined. It is the purpose of this paper to develop a method of setting up a Daniell integral which is applicable to a large group of cases, and in particular to functionals.†

2. **Definitions.** If K be any class, we shall define a *division* of K as a finite set of non-null sub-classes exhausting K at least once. Divisions of K will be represented by Greek letters with the suffix K—thus α_K, β_K, etc. A division depending on a parameter n will be represented by some such symbol as $\alpha_K(n)$. The sub-classes or intervals of a division α_K will be represented by $t_1(\alpha_K)$, $t_2(\alpha_K)$, $\cdots$, $t_m(\alpha_K)$.

A *weighted division* of K will be defined as a division of K to each term of which is assigned a positive number—its *weight*. A weighted division corresponding to α_k will be represented by such a symbol as α_K^U, α_K^V, or α_K^W. The weight of $t_j(\alpha_K)$ will be written $U_a\{t_j(\alpha_K)\}$, $V_a\{t_j(\alpha_K)\}$, or $W_a\{t_j(\alpha_K)\}$, respectively. $U_{a(n)}$ may be abbreviated to U_n if a specific sequence of α's is understood.

A K-*partition*‡ P_K is a sequence of weighted divisions $\alpha_K^{U_1}(1)$, $\alpha_K^{U_2}(2)$, $\cdots$, $\alpha_K^{U_n}(n)$, $\cdots$, such that

(1) Every member of $\alpha_K(n+1)$ is wholly included in one member of $\alpha_K(n)$ and one only.

(2) $U_n\{t_j(\alpha_K(n))\}$ is the sum of all the numbers $U_{n+1}\{t_l(\alpha_K(n+1))\}$ such that $t_l(\alpha_K(n+1))$ is contained in $t_j(\alpha_K(n))$.

(3) If S_n is the 'sum' of a number of intervals from $\alpha_K(n)$, whatever n may be, and if S_{n+1} is always entirely included in S_n, then either there is a K-element common to every S_n, or the sum of the weights of the intervals in S_n approaches 0 as n grows without limit.

* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), vol. 20 (1919).

† Cf. P. Lévy, Comptes Rendus, Aug., 1919.

‡ This notion is related to E. H. Moore's 'development.'

$\alpha_K(n)$, $\alpha_K^{U_n}(n)$, and $t_j(\alpha_K(n))$ are said to *belong* to P_k.

A function f defined for all elements of K is said to be a P_K *step-function* if there is an $\alpha_K(n)$ belonging to P_K such that the function is constant for every $t_j(\alpha_K(n))$. We shall say that f has degree n.

The *mean** of a P_K step-function f of degree n, taken over K with respect to P_K, is said to be

$$\frac{\sum_j U_n\{t_j(\alpha_K(n))\}f(x_j)}{\sum_j U_n\{t_j(\alpha_K(n))\}},$$

where x_j is a member of $t_j(\alpha_K(n))$. It will be written briefly $M_{P_K}(f)$.

A function f will be said to *have uniform P_K-continuity* if, given any positive number ϵ, there is an integer n such that if x and y belong to the same $t_j(\alpha_K(n))$, $|f(x) - f(y)| < \epsilon$.†

3. **Application of Daniell's Results.** Daniell's theory of integration is based on the existence of a set T_0 of bounded functions, which shall be closed with respect to multiplication by a constant, the addition of two functions, and the operation of taking the modulus. The class of all P_K step-functions clearly has all these properties. Daniell further postulates a finite functional operation I defined over T_0 and satisfying the conditions

(C) $I(cf) = cI(f)$, if c is any constant,

(A) $I(f_1 + f_2) = I(f_1) + I(f_2)$,

(P) $I(f) \geqq 0$ if $f(p) \geqq 0$ for all p,

(L) If $f_1 \geqq f_2 \geqq \cdots \geqq 0 = \lim f_n$ for every p,

$$\lim I(f_n) = 0.$$

Our operation M_{P_K} satisfies all these conditions. The first three need no proof, being matters merely of elementary algebra. (L) may be proved if we can show that for every positive number a, the total weight of the set of intervals containing the K-elements p for which $f_n(p) \geqq a$ approaches 0 as n grows indefinitely. Consider the set S_n of elements p for which $f_n(p) \geqq a$. Clearly S_{n+1} is included in S_n. Three conceivable possibilities are open.

* The use of mean instead of integral is found in the posthumous papers of Gateaux (Bull. de la Soc. Math. de France, 1919). This was however unknown to me at the time I wrote this article.

† If a term x is shared by two or more members of some $\alpha(n)$, in determining functions over K, we regard x as a set of different members of K, each consisting of x *qua* member of all of the intervals of a sequence $t_1(\alpha_K(1))$, $t_2(\alpha_K(2))$, $\cdots$, where $t_{n+1}(\alpha_K(n+1))$ is contained in $t_n(\alpha_K(n))$. Each $t_n(\alpha_K(x))$ will contain only this value of x. It may be shown (in the general case, with the aid of Zermelo's axiom) that this change will not affect the validity of the three conditions for a division, and will render every step-function single-valued.

(1) Every S_n contains intervals from some fixed $\alpha_K(m)$. In this case there is some term p common to every S_n, so that for every n, $f_n(p) \geqq a$. This however is contrary to our hypothesis.

(2) (1) is not satisfied, but there is some p common to every S_n. This again violates our hypothesis.

(3) Neither (1) nor (2) is satisfied. Then by (3) of the definition of a partition, the sum of the weights of the intervals in S_n approaches 0 as n grows without limit. In this case it follows from the definition of M_{P_K} that $\lim_{n=\infty} M_{P_K}(f_n) \leqq a$.

Hence (L) is proved, and we are at once in a position to apply Daniell's results. He defines T_1 as the class of all functions f which are the limits of a sequence f_n where every f_n belongs to T_0 and $f_1 \leqq f_2 \leqq \cdots$. $I(f)$ is defined as the limit of $I(f_n)$. Given any function f, $\bar{I}(f)$ is defined as the lower bound of $I(\varphi)$ for all functions φ of class T_1 such that $\varphi \geqq f$, and $\underline{I}(f)$ as $-\bar{I}(-f)$. If $\bar{I}(f) = \underline{I}(f)$ and is finite, f is said to be summable, and it is proved that if $f_1, f_2, \cdots, f_n, \cdots$, is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leqq \varphi$ for all n, f is summable, and $\lim I(f_n) = I(f)$. We can translate all this into our language, and in particular we can say that if a function can be obtained as a limit of a set of P_K step-functions all less in modulus than a certain P_K step-function, it is summable and its mean may be determined. A constant is clearly a P_K step-function, and we neither gain nor lose any generality by insisting that all our P_K step-functions be less in modulus than some constant.

Every bounded uniformly P_K-continuous function is summable. For let $f(x)$ be such a function and let $f_n(x)$ be the maximum of $f(x)$ in an interval $t_j(\alpha_K(n))$ that contains x and belongs to P_K. It is clear that by definition f_n is a P_K step-function, and furthermore that $|f_n(x)| < 1 + \max|f(x)|$. Moreover, since f is uniformly P_K-continuous, and since $f_n(x) = f(x_n)$ where x and x_n lie in the same $t_j(\alpha_K(n))$, $\lim_{n=\infty} |f(x) - f_n(x)| = 0$, so that $f(x) = \lim f_n(x)$.

An important generalization of this theorem is the following: *if L_m is the set of all K-elements contained in a finite number of intervals of P_K, if L_m is contained in L_{m+1} for all m and if every K-element is in some L_m, then if f is bounded and uniformly continuous over all the intervals of each L_m, it is summable according to M_{P_K}.* Let $g_m(x) = f(x)$ when x is in L_m and 0 otherwise. Every g_m is summable by the argument of the last paragraph; and the whole set is bounded as no g_m can be larger than f in its largest modulus. As f is the limit of $\{g_m\}$, by Daniell's theorem, f is summable.

4. **Examples.** (*a*) K is the set of all real numbers in the closed interval (a, b). $\alpha_K(n)$ is the set of intervals

$$\left\{\frac{a+h(b-a)}{2^n}, \quad \frac{a+(h+1)(b-a)}{2^n}\right\} \qquad h < 2^n$$

and each interval of $\alpha_K(n)$ is given the weight $1/2^n$. The mean $M_{P_K}(f)$ is the Lebesgue integral $1/(b-a)\int_a^b f(x)dx$, which is what we should naturally call the mean of $f(x)$ over the interval (a, b). This results immediately from the facts (1) that any continuous function can be obtained as the limit of a bounded sequence of step-functions constant over all the intervals of some $\alpha_K(n)$; (2) that, as Daniell has shown, every Lebesgue summable function is summable in his sense if $I(f)$ be Riemann integration and T be the set of all continuous functions, and hence if $I(f)$ be the Riemann integral confined to some set of functions with respect to which all continuous functions are summable.

(*b*) K is the set of all points $(x_1, x_2, \cdots, x_n)$ in a bounded region V of n-space of 'volume' v. $\alpha_K(l)$ is the set of all intervals

$$a_k/2^l \leq x_k \leq (a_k+1)/2^l \qquad (1 \leq k \leq n),$$

where the a_k's are integers ranging between bounds not less than the largest coördinate of any point in V. The weight of each interval of $\alpha_K(l)$ is the limit of the sum of the 'volumes' of the intervals of $\alpha_K(m)$ contained in this interval to the sum of the 'volumes' of all intervals of $\alpha_K(m)$, as m grows indefinitely, and V shall be such that this limit always exists. $M_{P_K}(f)$ becomes

$$(1/v)\int\int_V \cdots \int f(x_1, x_2, \cdots, x_n)dV.$$

(*c*) K is the set of all points $(x_1, x_2, \cdots, x_m, \cdots)$ in a region of space of a denumerably infinite number of dimensions such that $a_m \leq x_m \leq b_m$ for all m. $\alpha_K(n)$ is the set of regions

$$\frac{2^{n-m}a_m + h_m(b_m - a_m)}{2^{n-m}} \leq x_m \leq \frac{2^{n-m}a_m + (h_m+1)(b_m - a_m)}{2^{n-m}}$$

for all $m \leq n$. (Here h_m is some integer between 0 and $2^{n-m} - 1$). The weight of each region is

$$\frac{1}{2^{n(n+1)/2}}.$$

In the case where for every m $a_m = 0$ and $b_m = 1$, $M_{P_K}(f)$ is Daniell's

$$I(f) = \int_0^1 \cdots \int_0^1 \cdots f(p)dx_1dx_2 \cdots dx_n \cdots,$$

and is defined for a class of functions including all continuous functions. This results from the fact that every function belonging to Daniell's T_0, the class of all continuous functions of a finite number of variables, is summable in our system.

(*d*) K is the set of all continuous functions defined between 0 and 1, satisfying a Lipschitz condition and themselves lying between the bounded continuous functions $\varphi(x)$ and $\psi(x)$. The coefficient of irregularity of a function f (written $c(f)$) is the upper bound of the modulus of the slope of a chord of the curve representing the function. $\alpha_K(n)$ is the set of regions each of which consists of all continuous functions f satisfying simultaneously the 2^n pairs of inequalities

$$\frac{2^{2^n-m}\varphi\left(\frac{m}{2^n}\right)+h_m\left\{\psi\left(\frac{m}{2^n}\right)-\varphi\left(\frac{m}{2^n}\right)\right\}}{2^{2^n-m}} \le f\left(\frac{m}{2^n}\right)$$

$$\le \frac{2^{2^n-m}\varphi\left(\frac{m}{2^n}\right)+(h_m+1)\left\{\psi\left(\frac{m}{2^n}\right)-\varphi\left(\frac{m}{2^n}\right)\right\}}{2^{2^n-m}}$$

and an inequality either of the form

$$k_m \le c(f) \le k_m + 1 \qquad \text{or} \qquad c(f) \ge m$$

for all integral values of m not greater than 2^n. Here h_m is an integer between 0 and 2^{2^n-m}, and k_m is an integer between 0 and $m - 1$. The coefficient of irregularity of an interval is defined as the smallest integer not less than 2 that is greater than the coefficient of irregularity of some function in the interval. The weighting of $\alpha_K(n)$ is carried on in a progressive manner as follows: when K is divided into intervals of $\alpha_K(1)$ or when any interval of $\alpha_K(n)$ is divided into intervals of $\alpha_K(n+1)$, all intervals with the same coefficient of irregularity are weighted alike. The total weight of all the sub-intervals with a given coefficient of irregularity c greater than that of the original interval we shall then make $w/c!$, where w is the weight of the original interval. The rest of the weight, of course, goes to those sub-intervals with the same coefficient of irregularity as the original interval. If w is the original weight of K, and q the least coefficient of irregularity of any function it contains, the total weight of those intervals in any division whose coefficients of irregularity exceed k may be shown to be no greater than $w(e-2)^k$, if $k > q$.

That is, the total weight of any set of intervals whose coefficient of irregularity is greater than a given number N approaches 0 as N grows without limit. Furthermore, if $S_1, S_2, \cdots S_n, \cdots$ are sets of intervals such that every S_n contains intervals with a coefficient of irregularity not greater than a fixed number N, and if S_{n+1} is always contained in S_n,

there is always a continuous function belonging to every S_n. To prove this, in the first place, if we discard from S_n all intervals whose coefficient of irregularity equals or exceeds N, and call the remaining set R_n, R_{n+1} will be contained in R_n, which will always exist, whatever n. Let us divide the intervals of R_1 into those that contain intervals from an infinity of R_n's and those that do not. Clearly there will be at least one interval in the former class. Select one such interval.* This will contain intervals belonging to some R_k. These intervals can again be divided into two classes, and we can again select those that contain sub-intervals of an infinite sequence of orders. In this way an infinite chain $t_1, t_2, \cdots, t_n, \cdots$ can be selected of intervals belonging respectively to $S_1, S_k, \cdots, S_l, \cdots$, and all of a coefficient of irregularity no greater than N. From a certain stage on these t's will contain no function whose coefficient of irregularity is more than $N + 1$. By a theorem of Frechet,† since t_1 is a bounded class of equally continuous functions, it will be compact. It may readily be proved that every t_n is closed and hence extremal.‡ Consequently there will be a continuous function f common to every t_n, and therefore to every S_n. This completes the proof of (3) of the definition of a partition. The satisfaction of (1) and (2) is immediately obvious. We are hence in a position to apply our definition of a mean to functionals of continuous functions, and to give a meaning of $M_{P_{\overline{K}}}\{F\}$.

It should be noted that there is much that is arbitrary in the actual carrying out of this definition. What is really essential is that some coefficient of irregularity be chosen so that every "Einschachtelung" of intervals of less than a given coefficient of irregularity should contain a continuous function, and that then a method of weighting be adopted which shall make the total weight of the set of intervals whose coefficient of irregularity is greater than N a decreasing function of N that approaches 0 as a limit. This method can at once be extended to space-curves, surfaces, and all such entities as are usually made the arguments of functionals.

It clearly follows by the theorem at the end of § 3 that if a functional is bounded and is uniformly P_L-continuous over every P_L that consists of all the $\alpha_K(n)$'s restricted to functions of no more than a given coefficient of irregularity, it is summable with respect to M_{P_K}. Now, the set of all functions lying between two given functions in modulus and of no more than a given coefficient of irregularity is extremal, as may be proved from the fact that it is bounded and equally continuous. Hence by a theorem

* An ordinal arrangement of every $\alpha_K(n)$ can be found which will make this and the following selections perfectly determinate, and will consequently avoid the difficulties of Zermelo's axiom.

† M. Fréchet, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 37.

‡ Ibid., p. 7.

of Fréchet, every bounded functional F continuous over the set is uniformly continuous over the set in Fréchet's sense. This means that for every δ there is an ϵ independent of f and g such that if

$$|f(x) - g(x)| < \epsilon$$

for every x,

$$|F(f) - F(g)| < \delta$$

Every functional uniformly continuous in his sense is uniformly continuous in ours, for if two functions f and g have coefficients of irregularity less than or equal to N and if

$$|f(x) - g(x)| < \epsilon$$

for all the points between $x = 0$ and $x = 1$ for which $x = a/2^n$, then

$$|f(x) - g(x)| < \epsilon + \frac{Na}{2^{n-1}}$$

for all points. Since $\epsilon + Na/2^{n-1}$ becomes smaller and smaller as we constrain f and g to lie within smaller and smaller intervals of P_K, the dependence of our definition of uniform continuity on that of Fréchet follows. Hence every bounded continuous functional is summable in accordance with our definition.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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ON CERTAIN DETERMINANTS ASSOCIATED WITH TRANSFORMATIONS EMPLOYED IN THERMODYNAMICS.

By J. E. Trevor.

1. Introduction. When a fluid mixture of two distinct component substances is in a state of thermodynamic equilibrium, the energy $E(x_1, x_2, x_3, x_4)$ of the mixture is a positively homogeneous function, of degree one, of the volume x_1, the entropy x_2, and the component-masses x_3, x_4 of the mixture. If δE denotes the increment of E when the variables are given independent increments $\delta x_1, \cdots, \delta x_4$, it is concluded from the principles of thermodynamics that the equilibrium is stable when the sum of terms of order two in Taylor's expansion of δE is positive for all sets of sufficiently small increments that do not satisfy the conditions

$$\delta x_1/x_1 = \delta x_2/x_2 = \delta x_3/x_3 = \delta x_4/x_4.$$

Again, when the specific volume y_1, the specific entropy y_2, and the specific component-masses y_3, y_4 are defined by equations $y_i = x_i/(x_3 + x_4)$, the energy E is a function $(x_3 + x_4)\cdot E(y_1, y_2, y_3, 1 - y_3)$, i.e. a function $(x_3 + x_4)\cdot e(y_1, y_2, y_3)$; whereupon it is concluded* that stability is ensured when the sum of terms of order two in the expansion of δe is positive for all sufficiently small increments of y_1, y_2, y_3. From either of these two criteria of stability a set of necessary and sufficient conditions of stability may be deduced.

To obtain equivalent sets of conditions in other independent variables it is customary, on writing p_i for $\partial E/\partial x_i$, to employ the transformations

$$x_1' = p_1, \quad x_2' = x_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_1x_1;$$

$$x_1' = x_1, \quad x_2' = p_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_2x_2;$$

$$x_1' = p_1, \quad x_2' = p_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_1x_1 - p_2x_2;$$

and similar transformations with reference to y_1, y_2, y_3, e. The conditions obtained, and their immediate consequences, are inequalities giving the signs of the hessians of the functions E, E', e, and e', and of the principal minors of these hessians.

Now the elements of any one of these determinants include derivatives with regard to a of the variables p_i, where a may be zero. In seeking a

* J. E. Trevor, Amer. Math. Monthly, vol. 26, 444 (1919).

convenient rule for finding the sign of any determinant D of the set, I have observed that D is positive when a is even (0, 2, or 4), and is negative when a is odd; save that D vanishes identically when it is the determinant of the second derivatives, of a function E or E', with regard to all the x's that occur in the function. The exception asserts that when any of the p_i are held constant the others are not independent; which is obviously true, since the first derivatives of a homogeneous function of degree one are connected by a relation.

The observation that the signs of the determinants D are given by a simple rule has led me to seek a theorem of which the rule is a manifestation. The application of the result found, unlike that of the empirical rule, is restricted neither to a limited number of variables, to analytic functions, to positive values of the variables, nor to *principal* minors.

2. The Minors of the Hessians of Certain Related Functions. Let $e(x_1, x_2, \cdots, x_n)$ be a continuous function of the independent variables $x_1, x_2, \cdots, x_n$, with continuous first and second derivatives. Writing $p_i = \partial e/\partial x_i$, consider the transformations

$$x_1' = p_1, \; \cdots, \; x_m' = p_m, \qquad x_{m+1}' = x_{m+1}, \; \cdots, \; x_n' = x_n,$$

$$f_m = e - \sum_{i=1}^{m} p_i x_i \qquad (m = 1, 2, \cdots, n).$$

The jacobians of these transformations are the hessian of e and the principal minors of this hessian. When any of these jacobians vanish, the corresponding transformation is degenerate and shall be excluded from the set. The differentials of the functions $f_m(p_1, \cdots, p_m, x_{m+1}, \cdots, x_n)$ are

$$df_m = -\sum_{i=1}^{m} x_i dp_i + \sum_{j=m+1}^{n} p_j dx_j.$$

Let us now consider the set of $2n$ elements, in "normal order,"

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n;$$

and from this set form an arbitrary combination C, in "normal order," by suppressing any n elements. As an illustration let us choose the set for which $m = 4$ and $n = 9$, and from it form the combination

$$p_1x_1 \quad {*x_2} \quad p_3{*} \quad p_4x_4 \quad {**} \quad p_6{*} \quad {*x_7} \quad p_8{*} \quad {**},$$

where the suppressions are indicated by asterisks. Terming the letters of the sets $x_1, \cdots, x_m$ and $p_1, \cdots, p_m$ respectively the x_{I} and the p_{I}, and terming the remaining x's and p's respectively the x_{II} and the p_{II}, we proceed to tabulate the indices of the elements of each set "present in" and "absent from" the combination C.

	p_1x_1	$*x_2$	p_3*	p_4x_4		$**$	p_6*	$*x_7$	p_8*	$**$
Present x_{I}	1	2		4	Present x_{II}			7		
Absent x_{I}			3		Absent x_{II}	5	6		8	9
Present p_{I}	1		3	4	Present p_{II}		6		8	
Absent p_{I}		2			Absent p_{II}	5		7		9

On forming the jacobian of the letters of C, with regard to the variables x_{I}, x_{II}, hereby denoting $\partial^2 e/\partial x_i \partial x_j$ by e_{ij}; and on forming the jacobian of these letters with regard to the variables p_{I}, x_{II}, hereby denoting the second derivative of f_m, i.e. of f_4, with regard to its ith and jth variables by f_{ij}; we obtain,

$$\begin{vmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} & e_{46} & e_{47} & e_{48} & e_{49} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ e_{61} & e_{62} & e_{63} & e_{64} & e_{65} & e_{66} & e_{67} & e_{68} & e_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ e_{81} & e_{82} & e_{83} & e_{84} & e_{85} & e_{86} & e_{87} & e_{88} & e_{89} \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_{11} & -f_{12} & -f_{13} & -f_{14} & -f_{15} & -f_{16} & -f_{17} & -f_{18} & -f_{19} \\ -f_{21} & -f_{22} & -f_{23} & -f_{24} & -f_{25} & -f_{26} & -f_{27} & -f_{28} & -f_{29} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -f_{41} & -f_{42} & -f_{43} & -f_{44} & -f_{45} & -f_{46} & -f_{47} & -f_{48} & -f_{49} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} & f_{67} & f_{68} & f_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ f_{81} & f_{82} & f_{83} & f_{84} & f_{85} & f_{86} & f_{87} & f_{88} & f_{89} \end{vmatrix}.$$

By inspection we observe that the first of these jacobians (apart from its sign) is obtainable from the hessian of $e(x_{\mathrm{I}}, x_{\mathrm{II}})$ by deleting rows and columns as follows.

Illustrative Case		In the General Case
Delete columns	124, 7	Delete columns of the present x_{I} and x_{II}.
Retain rows	134, 68	Retain rows of the present p_{I} and p_{II};
I.e., delete rows	2, 579	I.e., delete rows of the absent p_{I} and p_{II}.

We observe similarly that the second jacobian (apart from its sign) is obtainable from the hessian of $f_m(p_{\mathrm{I}}, x_{\mathrm{II}})$ by deleting rows and columns as follows.

Illustrative Case		In the General Case
Delete columns	134, 7	Delete columns of the present p_{I} and x_{II}.
Retain rows	124, 68	Retain rows of the present x_{I} and p_{II};
I.e., delete rows	3, 579	I.e., delete rows of the absent x_{I} and p_{II}.

If we denote the minor of the hessian of a function u, obtained by deleting the rows $i \cdots k$ and the columns $j \cdots l$, by the symbol $U\binom{i \cdots k}{j \cdots l}$, the above jacobians are denoted by

$$\sigma_0 \cdot E\begin{pmatrix} & 2, & 579 \\ 124, & & 7 \end{pmatrix}, \qquad \sigma_1 \cdot F_4\begin{pmatrix} & 3, & 579 \\ 134, & & 7 \end{pmatrix};$$

where the sign-factors σ_0, σ_1 are yet undetermined. From the above tabulations, the rule for the formulation of the jacobians of the n letters of any combination C, for any values of n and m, is expressed by

$$\sigma_0 \cdot E \begin{pmatrix} \text{Indices of the absent } p_{\text{I}} \text{ and } p_{\text{II}} \\ \text{Indices of the present } x_{\text{I}} \text{ and } x_{\text{II}} \end{pmatrix},$$

and

$$\sigma_1 \cdot F_m \begin{pmatrix} \text{Indices of the absent } x_{\text{I}} \text{ and } p_{\text{II}} \\ \text{Indices of the present } p_{\text{I}} \text{ and } x_{\text{II}} \end{pmatrix};$$

or, more generally by

$$\sigma \cdot U \begin{pmatrix} \text{Indices of the first derivatives of } u \text{ absent from } C \\ \text{Indices of the independent variables of } u \text{ present in } C \end{pmatrix} \quad (u = e, f_m).$$

In seeking the sign-factor σ_0, we consecutively number the places of the n letters in the illustrative combination C,

Combination,	p_1	x_1	x_2	p_3	p_4	x_4	p_6	x_7	p_8
Place-numbers,	1	2	3	4	5	6	7	8	9

and count the number λ_0 of transpositions of letters necessary to bring each of the variables x_i that is present in C to the ith place in C, without disturbing the order of the p_i that occur in C. These transpositions bring the letters of C into the arrangement

$$\begin{matrix} x_1 & x_2 & p_1 & x_4 & p_3 & p_4 & x_7 & p_6 & p_8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

The corresponding transpositions of rows, in the jacobian of the letters of C with regard to the variables x_{I}, x_{II}, bring all the elements 1 of the array on to the principal diagonal. Hence the sign of the jacobian is that of $\sigma_0 = (-1)^{\lambda_0}$. The number λ_0 can indeed have different values. But these are either all even or all odd.

In seeking the sign-factor σ_1, we count the number λ_1 of transpositions necessary to bring each of the variables y_i that is present in C, where $y_i = p_{\text{I}}, x_{\text{II}}$, to the ith place in C, without disturbing the order of the x_{I} and p_{II} that occur in C. These transpositions bring the letters of C into the arrangement

$$\begin{matrix} p_1 & x_1 & p_3 & p_4 & x_2 & x_4 & x_7 & p_6 & p_8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

The corresponding transpositions of rows, in the jacobian of the letters of C with regard to the variables p_{I}, x_{II}, bring all the elements 1 of the array on to the principal diagonal. Hence the sign of the jacobian of the letters of the combination is that of $(-1)^{\lambda_1}$. Now the jacobian of the

letters of C, with regard to the variables p_{I}, x_{II}, contains the $\mu = 3$ rows (124) of negative elements corresponding to the μ x_{I}'s that are present in C. On factoring out -1 from each of these rows, the determinant to which the jacobian reduces (on deleting the rows and columns of the elements 1, and on taking out the factors -1) is multiplied by

$$\sigma_1 = (-1)^{\lambda_1+\mu}.$$

So, in general, the jacobian of the letters of the combination C, in "normal order," of any n letters of the set

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n,$$

with regard to the variables x_{I}, x_{II}, is a minor of the hessian of $e(x_{\mathrm{I}}, x_{\mathrm{II}})$,

$$(-1)^{\lambda}\cdot E\begin{pmatrix}\text{Indices of the first derivatives of } e \text{ absent from } C\\ \text{Indices of the independent variables of } e \text{ present in } C\end{pmatrix};$$

and the jacobian of the letters of the combination, with regard to the variables p_{I}, x_{II}, is a minor of the hessian of $f_m(p_{\mathrm{I}}, x_{\mathrm{II}})$,

$$(-1)^{\lambda+\mu}\cdot F_m\begin{pmatrix}\text{Indices of the first derivatives of } f_m \text{ absent from } C\\ \text{Indices of the independent variables of } f_m \text{ present in } C\end{pmatrix}.$$

In each of these formulations the numbers λ and μ are counted with reference to the combination and to the variables taken independent; λ being the number of transpositions necessary to bring each of the independent variables to the place of its index in C, and μ being the number of the variables x_{I} present in C.

3. A Matrix of the Hessians and their Minors. It is now proposed to arrange in a column the combinations C, in "normal order," of the elements

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n,$$

taken n at a time; and then to tabulate the jacobians of the letters of each combination, with regard to the variables of the successive sets

$$x_1, x_2, x_3, \cdots, x_n, \qquad p_1, x_2, x_3, \cdots, x_n, \qquad p_1, p_2, x_3, \cdots, x_n,$$
$$\cdots, \qquad p_1, p_2, p_3, \cdots, p_n;$$

thereby expressing the jacobians as minors, of all orders from zero to n, of the respective hessians $E()$, $F_1()$, $\cdots$, $F_n()$.

In this tabulation the successive places of the row corresponding to any given combination will be occupied by elements

$$(-1)^{\lambda}\cdot E(\vdots\vdots), \qquad (-1)^{\lambda+\mu}\cdot F_m(\vdots\vdots), \qquad (m = 1, 2, \cdots, n)$$

where λ, μ are counted in each case with reference to the combination

and to the variables taken independent. The tabulated array of minors has $(2n)!/(n!)^2$ rows, one for each combination C; and it has $n+1$ columns, one for each of the successive sets of variables. Hence the number of the elements constituting the array is $(2n)!(n+1)/(n!)^2$. For $n = 1$ this number is 4, for $n = 2$ it is 18, for $n = 3$ it is 80, and for $n = 4$ it is 350.

Let the ith element $(-1)^{\lambda}\cdot E(\because)$ of the first column of the array be denoted by E_i. Then, because of the relation

$$\frac{\partial(C_i \quad)}{\partial(x_1, x_{11})} = \frac{\partial(C_i \quad)}{\partial(p_1, x_{11})}\frac{\partial(p_1, x_{11})}{\partial(x_1, x_{11})},$$

where the symbol C_i represents the letters of the ith combination, the $(m+1)$th element of the ith row satisfies the equation

$$E_i = (-1)^{\lambda-\mu}\cdot F_m(\because) \times E\begin{pmatrix} m+1, \cdots, n \\ m+1, \cdots, n \end{pmatrix},$$

or

$$(-1)^{\lambda+\mu}\cdot F_m(\because) = \left[E\begin{pmatrix} m+1, \cdots, n \\ m+1, \cdots, n \end{pmatrix}\right]^{-1}\cdot E_i = a_m E_i.$$

Thus each element of the $(m+1)$th column of the array differs from the element E_i of the same row by the same factor a_m. We find that the array forms a matrix such that all determinants of orders greater than one that can be formed from it are equal to zero. The matrix is of rank one. The factors a_m are the reciprocals of the hessian $E()$ and of the principal minors obtained by successively deleting the last $1, 2, \cdots, n-1$ rows and columns of $E()$. The factor a_1 is the reciprocal of $\partial^2 e/\partial x_1^2$. The conclusion that the hessians of the functions e and f_m, and the minors of these hessians, are connected in this way is the theorem sought.

To find the combination C that corresponds to any minor

$$\pm U\begin{pmatrix} \text{Indices of the first derivatives of } u \text{ absent from } C \\ \text{Indices of the independent variables of } u \text{ present in } C \end{pmatrix} \quad (u = e, f_m),$$

of any order from zero to n, of the hessian of any function e or f_m, we have the rule,

Comparing the symbol for the minor with the list of the first derivatives and of the independent variables of u, write the derivatives not absent and the variables present, and arrange them in normal order.

For example let $F_2(^{23}_{13})$ be given, for $n = 3$. The derivatives are $-x_1$, $-x_2$, p_3, and the variables are p_1, p_2, x_3. The derivative not absent is $-x_1$, and the variables present are p_1, x_3. Hence $C = p_1x_1x_3$. Again, let $F_3(^{2}_{2})$ be given, for $n = 4$. Derivatives are $-x_1$, $-x_2$, $-x_3$, p_4, and variables are p_1, p_2, p_3, x_4. Derivatives not absent are $-x_1$, $-x_3$, p_4, and the variable present is p_2. Hence $C = x_1p_2x_3p_4$.

In the case of a principal minor, which is formed by deleting the same columns as rows, the indices of the derivatives *not absent* and those of the variables *present* form a set of indices 1, 2, $\cdots$, n. Hence no transpositions are necessary to bring the ith variable to the ith place in C,—for principal minors $\lambda = 0$. In particular the sign-factors of the principal minors $(-1)^\lambda \cdot E(:::)$ of $E()$ are all $+1$.

The properties of the matrix of the hessian minors immediately yield a generalization of the empirical rule stated in the introduction. Let it be given that the hessian $E()$ and all its principal minors are positive. Then any element of the rows of the principal minors (of all orders) of the $F_m()$, since it is equal to a ratio of two elements of the column of principal minors of $E()$, is equal to a positive quantity. Hence any principal minor of the $F_m()$ has the sign of its sign-factor, and so is positive or negative according as its $\lambda + \mu$ is even or odd. But $\lambda = 0$ for principal minors. So any principal minor of $F_m()$ is negative when and only when μ is odd. Now μ is the number of the variables $x_1, x_2, \cdots, x_m$ appearing in the corresponding combination C, and can have any of the values 0, 1, 2, $\cdots$, m. Any variable x_i of the set appears in C when and only when the corresponding numeral i is missing from the upper row of indices in the symbol $F_m(:::)$ for the minor in question. When none are missing $\mu = 0$. So the minor is negative when and only when any odd number of the first m row-and-column pairs is not deleted; i.e., when the elements of the array of the minor include derivatives with regard to an odd number of the variables $p_1, p_2, \cdots, p_m$. Further, when the function e is homogeneous of degree one, its first derivatives are homogeneous functions of degree zero, wherefore the jacobian $E()$ of these derivatives vanishes identically. This causes the column of the minors of $F_n()$ to disappear from the matrix, and it causes all the minors in the row of the element $E()$ to vanish. These minors are the determinants of the second derivatives with regard to all the x's that occur in the respective functions f_m. The results thus obtained constitute a generalization of the empirical rule for the signs of the principal minors of the hessians $F_m()$. And it may be added that the elements of any row of secondary minors (including their sign-factors) in the matrix have the same sign, and vanish together.

Further cases of particular interest arise when one or more of the principal minors of $E()$ vanish identically. When m has a particular value c, then the $(c+1)$th column of the matrix is the column of the minors of $F_c()$, and the factor a_c for this column is the reciprocal of the determinant

(1) $$E\begin{pmatrix} c+1, \cdots, n \\ c+1, \cdots, n \end{pmatrix}.$$

If this determinant vanishes identically, which it will do if the primitive function $e(x_1, x_2, \cdots, x_n)$ is homogeneous of degree one in $x_1, x_2, \cdots, x_c$, then the transformation

$$x_1' = p_1, \quad \cdots, \quad x_c' = p_c, \qquad x_{c+1}' = x_{c+1}, \quad \cdots, \quad x_n' = x_n,$$

$$f_c = e - \sum_{i=1}^{c} p_i x_i$$

is degenerate, wherefore the $(c + 1)$th column disappears from the matrix, and we have that the elements of the row of (1), i.e., the elements of the row of the combination

$$C = p_1 p_2 \cdots p_c x_{c+1} x_{c+2} \cdots x_n$$

vanish identically. The special case $c = n$ was considered in the preceding paragraph, where the condition $E(\ ) = 0$ appeared as a consequence of the circumstance that the primitive function e was a homogeneous function of degree one.

4. Special Cases of the Matrix. When $n = 1$ we have

$$de = pdx, \qquad f = e - px, \qquad df = -xdp.$$

The $2n$ elements are p, x, the two combinations C are p and x, and we have $\lambda = \mu = 0$ for all entries. The matrix asserts merely that

$$\frac{1}{d^2e/dx^2} = \frac{-d^2f/dp^2}{1},$$

i.e., that $(dp/dx)^{-1} = dx/dp$.

When $n = 2$, the $2n$ elements are p_1, x_1, p_2, x_2. Writing the six combinations C in a column, entering the values of the λ and μ, and arranging the jacobians of the letters of each combination with regard to the variables of the successive sets of independent variables, we obtain the following tabulation.

C	λ	$\frac{\partial(C\ )}{\partial(x_1, x_2)}$	λ	μ	$\frac{\partial(C\ )}{\partial(p_1, x_2)}$	λ	μ	$\frac{\partial(C\ )}{\partial(p_1, p_2)}$
p_1x_1	1	$-E\binom{2}{1}$	0	1	$-F_1\binom{2}{1}$	0	1	$-F_2\binom{2}{1}$
p_1p_2	0	$+E(\ )$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$
p_1x_2	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$
x_1p_2	0	$+E\binom{1}{1}$	0	1	$-F_1(\ )$	0	1	$-F_2\binom{2}{2}$
x_1x_2	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\ )$
p_2x_2	0	$+E\binom{1}{2}$	0	0	$+F_1\binom{1}{2}$	1	1	$+F_2\binom{1}{2}$

The tabulated minors of order zero are equal to unity. For when the independent variables are v, w, and $C = vw$, we have

$$U\binom{12}{12} = \partial(v, w)/\partial(v, w) = +1 \qquad (U = E, F_m).$$

So, in more conventional notation, writing Δ, Δ_m for the hessians of e, f_m, the above matrix becomes:

$$\left\| \begin{matrix} -\dfrac{\partial^2 e}{\partial x_1 \partial x_2} & -\dfrac{\partial^2 f_1}{\partial p_1 \partial x_2} & -\dfrac{\partial^2 f_2}{\partial p_1 \partial p_2} \\ \Delta & \dfrac{\partial^2 f_1}{\partial x_2^2} & 1 \\ \dfrac{\partial^2 e}{\partial x_1^2} & 1 & -\dfrac{\partial^2 f_2}{\partial p_2^2} \\ \dfrac{\partial^2 e}{\partial x_2^2} & -\Delta_1 & -\dfrac{\partial^2 f_2}{\partial p_1^2} \\ 1 & -\dfrac{\partial^2 f_1}{\partial p_1^2} & \Delta_2 \\ \dfrac{\partial^2 e}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f_1}{\partial x_2 \partial p_1} & \dfrac{\partial^2 f_2}{\partial p_2 \partial p_1} \end{matrix} \right\|.$$

In this matrix each element of the second column differs from the corresponding element of the first by the factor $(\partial^2 e/\partial x_1^2)^{-1}$, and each element of the third column differs from the corresponding element of the first by the factor Δ^{-1}. We thus have,

$$\frac{1}{\partial^2 e/\partial x_1^2} = \frac{-\partial^2 f_1/\partial p_1^2}{1} = \frac{\partial^2 f_1/\partial x_2^2}{\Delta} = \frac{-\Delta_1}{\partial^2 e/\partial x_2^2} = \frac{\partial^2 f_1/\partial p_1 \partial x_2}{\partial^2 e/\partial x_1 \partial x_2},$$

$$\frac{1}{\Delta} = \frac{-\partial^2 f_2/\partial p_1^2}{\partial^2 e/\partial x_2^2} = \frac{-\partial^2 f_2/\partial p_2^2}{\partial^2 e/\partial x_1^2} = \frac{\Delta_2}{1} = \frac{\partial^2 f_2/\partial p_1 \partial p_2}{\partial^2 e/\partial x_1 \partial x_2}.$$

If the hessian Δ and its principal minors are positive, it follows, in accordance with the empirical rule, that

$$\partial^2 f_1/\partial p_1^2 < 0, \qquad \partial^2 f_1/\partial x_2^2 > 0, \qquad \Delta_1 < 0;$$

$$\partial^2 f_2/\partial p_1^2 < 0, \qquad \partial^2 f_2/\partial p_2^2 < 0, \qquad \Delta_2 > 0.$$

It follows also that the secondary minors,

$$\partial^2 e/\partial x_1 \partial x_2, \qquad \partial^2 f_1/\partial p_1 \partial x_2, \qquad \partial^2 f_2/\partial p_1 \partial p_2,$$

either have the same sign or vanish together.

If Δ vanishes, while its principal minors are positive, the function $f_2(p_1, p_2)$ disappears from the set of functions f_m, and we have

$$\partial^2 f_1/\partial p_1^2 < 0, \qquad \partial^2 f_1/\partial x_2^2 = 0, \qquad \Delta_1 < 0,$$

again according to rule. Here also $\partial^2 c/\partial x_1 \partial x_2$ and $\partial^2 f_1/\partial p_1 \partial x_2$ either have the same sign or vanish together.

C	λ	$\frac{\partial(C)}{\partial(x_1, x_2, x_3)}$	λ	μ	$\frac{\partial(C)}{\partial(p_1, x_2, x_3)}$	λ	μ	$\frac{\partial(C)}{\partial(p_1, p_2, x_3)}$	λ	μ	$\frac{\partial(C)}{\partial(p_1, p_2, p_3)}$
$p_1x_1p_2$	1	$-E\binom{3}{1}$	0	1	$-F_1\binom{3}{1}$	1	1	$+F_2\binom{23}{12}$	1	1	$+F_3\binom{23}{12}$
$p_1x_1x_2$	2	$+E\binom{23}{12}$	1	1	$+F_1\binom{23}{12}$	0	2	$+F_2\binom{3}{1}$	0	2	$+F_3\binom{3}{1}$
$p_1x_1p_3$	1	$-E\binom{2}{1}$	0	1	$-F_1\binom{2}{1}$	0	1	$-F_2\binom{2}{1}$	0	1	$-F_3\binom{23}{13}$
$p_1x_1x_3$	1	$-E\binom{23}{13}$	0	1	$-F_1\binom{23}{13}$	0	1	$-F_2\binom{23}{13}$	0	2	$+F_3\binom{2}{1}$
$p_1p_2x_2$	1	$-E\binom{3}{2}$	1	0	$-F_1\binom{13}{12}$	0	1	$-F_2\binom{13}{12}$	0	1	$-F_3\binom{13}{12}$
$p_1p_2p_3$	0	$+E(\ )$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$	0	0	$+F_3\binom{123}{123}$
$p_1p_2x_3$	0	$+E\binom{3}{3}$	0	0	$+F_1\binom{13}{13}$	0	0	$+F_2\binom{123}{123}$	0	1	$-F_3\binom{12}{12}$
$p_1x_2p_3$	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$	0	1	$-F_3\binom{13}{13}$
$p_1x_2x_3$	0	$+E\binom{23}{23}$	0	0	$+F_1\binom{123}{123}$	0	1	$-F_2\binom{13}{13}$	0	2	$+F_3\binom{1}{1}$
$^*p_1p_3x_3$	0	$+E\binom{2}{3}$	0	0	$+F_1\binom{12}{13}$	0	0	$+F_2\binom{12}{13}$	1	1	$+F_3\binom{12}{13}$
$x_1p_2x_2$	1	$-E\binom{13}{12}$	1	1	$+F_1\binom{3}{2}$	0	2	$+F_2\binom{3}{2}$	0	2	$+F_3\binom{3}{2}$
$x_1p_2p_3$	0	$+E\binom{1}{1}$	0	1	$-F_1(\ )$	0	1	$-F_2\binom{2}{2}$	0	1	$-F_3\binom{23}{23}$
$x_1p_2x_3$	0	$+E\binom{13}{13}$	0	1	$-F_1\binom{3}{3}$	0	1	$-F_2\binom{23}{23}$	0	2	$+F_3\binom{2}{2}$
$x_1x_2p_3$	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\ )$	0	2	$+F_3\binom{3}{3}$
$x_1x_2x_3$	0	$+E\binom{123}{123}$	0	1	$-F_1\binom{23}{23}$	0	2	$+F_2\binom{3}{3}$	0	3	$-F_3(\ )$
$^*x_1p_3x_3$	0	$+E\binom{12}{13}$	0	1	$-F_1\binom{2}{3}$	0	1	$-F_2\binom{2}{3}$	1	2	$-F_3\binom{2}{3}$
$^*p_2x_2p_3$	0	$+E\binom{1}{2}$	0	0	$+F_1\binom{1}{2}$	1	1	$+F_2\binom{1}{2}$	1	1	$+F_3\binom{13}{23}$
$^*p_2x_2x_3$	0	$+E\binom{13}{23}$	0	0	$+F_1\binom{13}{23}$	1	1	$+F_2\binom{13}{23}$	1	2	$-F_3\binom{1}{2}$
$^*p_2p_3x_3$	0	$+E\binom{1}{3}$	0	0	$+F_1\binom{1}{3}$	1	0	$-F_2\binom{12}{23}$	2	1	$-F_3\binom{12}{23}$
$^*x_2p_3x_3$	1	$-E\binom{12}{23}$	1	0	$-F_1\binom{12}{23}$	0	1	$-F_2\binom{1}{3}$	1	2	$-F_3\binom{1}{3}$

It may be remarked that a recently described classification* of the elements of a certain set of derivatives, which is of consequence in thermodynamics, is essentially an application of the properties of the above matrix for $n = 2$.

When $n = 3$ the $2n$ elements are p_1, x_1, p_2, x_2, p_3, x_3, and the matrix of 20×4 elements is as on page 82.

The factors that convert the elements of the first column of this matrix into those of the second, the third, and the fourth are the reciprocals of the principal minors

$$E\binom{23}{23}, \qquad E\binom{3}{3}, \qquad E(\ ).$$

We thus have, arranging principal minors and secondary minors on separate lines, and omitting duplicate secondary minors (of the starred rows in the tabulation),

$$\frac{F_1\binom{1}{1}}{E(\ )} = \frac{F_1\binom{13}{13}}{E\binom{3}{3}} = \frac{F_1\binom{12}{12}}{E\binom{2}{2}} = \frac{1}{E\binom{23}{23}} = \frac{-F_1(\ )}{E\binom{1}{1}} = \frac{-F_1\binom{3}{3}}{E\binom{13}{13}} = \frac{-F_1\binom{2}{2}}{E\binom{12}{12}} = \frac{-F_1\binom{23}{23}}{1}$$

$$= \frac{F_1\binom{3}{1}}{E\binom{3}{1}} = \frac{F_1\binom{23}{12}}{E\binom{23}{12}} = \frac{F_1\binom{2}{1}}{E\binom{2}{1}} = \frac{F_1\binom{23}{13}}{E\binom{23}{13}} = \frac{F_1\binom{13}{12}}{E\binom{3}{2}} = \frac{-F_1\binom{3}{2}}{E\binom{13}{12}},$$

together with a similar set of equations for the minors of $F_2()$, and a third set for the minors of $F_3()$.

If the hessian $E()$ and its principal minors are positive, it follows, in accordance with the empirical rule, that the principal minors of the $F_m()$ have the signs prefixed to them in the table. It follows also that the sign of any secondary minor of any $F_m()$ is determined by the sign of the secondary minor of $E()$ in the same row, and that all the minors of the row vanish together.

If $E()$ vanishes, while its principal minors are positive, the function f_3 disappears. But the signs of the minors of $F_1()$ and $F_2()$ are determined as before, save that

$$F_1\binom{1}{1} = F_2\binom{12}{12} = 0,$$

which accords with the rule.

In constructing the matrix for any value of n, the symbols $F_m(:::)$ of any row may be rapidly obtained from the symbol $E(:::)$ of the same row by the rule,—Replace E by F_m, delete all *common* indices $1, \cdots, m$, and insert all *missing* indices $m + 1, \cdots, n$.

In the case $n = 4$, where the operation of the empirical rule was first observed, the matrix of the *principal* minors is as follows.

* J. E. Trevor, Amer. Math. Monthly, vol. 27, p. 258 (1920).

C	λ	$\frac{\partial(C\quad)}{\partial(x_1, x_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, x_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2, p_3, x_4)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2, p_3, p_4)}$
$x_1x_2x_3x_4$	0	$+E\binom{1234}{1234}$	0	1	$-F_1\binom{234}{234}$	0	2	$+F_2\binom{34}{34}$	0	3	$-F_3\binom{4}{4}$	0	4	$+F_4(\;)$
$p_1x_2x_3x_4$	0	$+E\binom{234}{234}$	0	0	$+F_1\binom{1234}{1234}$	0	1	$-F_2\binom{134}{134}$	0	2	$+F_3\binom{14}{14}$	0	3	$-F_4\binom{1}{1}$
$x_1p_2x_3x_4$	0	$+E\binom{134}{134}$	0	1	$-F_1\binom{34}{34}$	0	1	$-F_2\binom{234}{234}$	0	2	$+F_3\binom{24}{24}$	0	3	$-F_4\binom{2}{2}$
$x_1x_2p_3x_4$	0	$+E\binom{124}{124}$	0	1	$-F_1\binom{24}{24}$	0	2	$+F_2\binom{4}{4}$	0	2	$+F_3\binom{34}{34}$	0	3	$-F_4\binom{3}{3}$
$x_1x_2x_3p_4$	0	$+E\binom{123}{123}$	0	1	$-F_1\binom{23}{23}$	0	2	$+F_2\binom{3}{3}$	0	3	$-F_3(\;)$	0	3	$-F_4\binom{4}{4}$
$p_1p_2x_3x_4$	0	$+E\binom{34}{34}$	0	0	$+F_1\binom{134}{134}$	0	0	$+F_2\binom{1234}{1234}$	0	1	$-F_3\binom{124}{124}$	0	2	$+F_4\binom{12}{12}$
$p_1x_2p_3x_4$	0	$+E\binom{24}{24}$	0	0	$+F_1\binom{124}{124}$	0	1	$-F_2\binom{14}{14}$	0	1	$-F_3\binom{134}{134}$	0	2	$+F_4\binom{13}{13}$
$x_1p_2p_3x_4$	0	$+E\binom{14}{14}$	0	1	$-F_1\binom{4}{4}$	0	1	$-F_2\binom{24}{24}$	0	1	$-F_3\binom{234}{234}$	0	2	$+F_4\binom{23}{23}$
$p_1x_2x_3p_4$	0	$+E\binom{23}{23}$	0	0	$+F_1\binom{123}{123}$	0	1	$-F_2\binom{13}{13}$	0	2	$+F_3\binom{1}{1}$	0	2	$+F_4\binom{14}{14}$
$x_1p_2x_3p_4$	0	$+E\binom{13}{13}$	0	1	$-F_1\binom{3}{3}$	0	1	$-F_2\binom{23}{23}$	0	2	$+F_3\binom{2}{2}$	0	2	$+F_4\binom{24}{24}$
$x_1x_2p_3p_4$	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\;)$	0	2	$+F_3\binom{3}{3}$	0	2	$+F_4\binom{34}{34}$
$p_1p_2p_3x_4$	0	$+E\binom{4}{4}$	0	0	$+F_1\binom{14}{14}$	0	0	$+F_2\binom{124}{124}$	0	0	$+F_3\binom{1234}{1234}$	0	1	$-F_4\binom{123}{123}$
$p_1p_2x_3p_4$	0	$+E\binom{3}{3}$	0	0	$+F_1\binom{13}{13}$	0	0	$+F_2\binom{123}{123}$	0	1	$-F_3\binom{12}{12}$	0	1	$-F_4\binom{124}{124}$
$p_1x_2p_3p_4$	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$	0	1	$-F_3\binom{13}{13}$	0	1	$-F_4\binom{134}{134}$
$x_1p_2p_3p_4$	0	$+E\binom{1}{1}$	0	1	$-F_1(\;)$	0	1	$-F_2\binom{2}{2}$	0	1	$-F_3\binom{23}{23}$	0	1	$-F_4\binom{234}{234}$
$p_1p_2p_3p_4$	0	$+E(\;)$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$	0	0	$+F_3\binom{123}{123}$	0	0	$+F_4\binom{1234}{1234}$

When $E()$ and its principal minors are positive we have that eight of the principal minors of each of the $F_m()$ are negative, while the other eight are positive. If $E()$ vanishes, while its principal minors remain positive, the last column disappears from the table, and the signs are distributed as before save that

$$F_1\binom{1}{1} = F_2\binom{12}{12} = F_3\binom{123}{123} = 0;$$

all of which accords with the rule. The whole matrix of 350 elements is too extensive to quote.

CORNELL UNIVERSITY,
June, 1920.

THE PERMANENT GRAVITATIONAL FIELD IN THE EINSTEIN THEORY.

By Luther Pfahler Eisenhart.

1. In accordance with the theory of Einstein a permanent gravitational field is defined by a quadratic differential form

$$(1) \qquad ds^2 = \sum_{i,k}^{1,\ldots,4} g_{ik} dx_i dx_k \qquad (g_{ik} = g_{ki}),$$

where the g's, called the potentials of the field, are determined by the condition of satisfying ten partial differential equations of the second order, $G_{ik} = 0$. When the four coördinates x_i are functions of a single parameter, the locus of the point with these coördinates is a curve in four-space. If these functions are of such a character that the integral

$$(2) \qquad \int \sqrt{\Sigma g_{ik} dx_i dx_k}$$

is stationary along the curve, the curve is called a "world-line," or a geodesic, in the four-space.

Einstein* considered the case when x_1, x_2, x_3, are rectangular coördinates and x_4 represents the time, and assumed that the field was produced by a mass at the origin which did not vary with the time. In order to obtain the equations of the world-lines in the form which enabled him to establish his well-known expression for the precession of the perihelion of Mercury, Einstein made also the following assumptions:

A. The quantities g are independent of t.

B. The equations $g_{i4} = 0$ for $i = 1, 2, 3$.

C. The solution is spacially symmetric with respect to the origin of coördinates in the sense that the solution is unaltered by an orthogonal transformation of x_1, x_2, x_3.

D. At infinity the quantities $g_{ik} = 0$ for $i \neq k$, and

$$g_{44} = -g_{11} = -g_{22} = -g_{33} = 1.$$

Schwarzschild† using the first three of these assumptions and certain others integrated the equations $G_{ik} = 0$, and obtained (1) in the form

* Berlin Sitzungsberichte, 1915, p. 831.

† Berlin Sitzungsberichte, 1916, p. 189.

$$ds^2 = c^2\left(1 - \frac{\alpha}{R}\right)dt^2 - \frac{dR^2}{1 - \dfrac{\alpha}{R}} - R^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

where α is a constant depending on the mass at the origin. Levi-Civita* has given three solutions of the equations $G_{ik} = 0$, one of which includes the above, and Weyl† has given still another solution. Later Kottler‡ obtained the form (3) not by the solution of the equations $G_{ik} = 0$ but as a consequence of certain postulates. It is the purpose of this paper to accomplish the same result by the following set of postulates:

I. Assumptions A and B of Einstein, in accordance with which we write (1) in the form

$$ds^2 = V^2dt^2 - ds_0^2, \tag{4}$$

where

$$ds_0^2 = \sum_{i,k}^{1,2,3} a_{ik}dx_idx_k, \tag{5}$$

the functions V and a_{ik} being independent of t.

II. The function V is a solution of

$$\Delta_2\theta = 0,$$

where $\Delta_2\theta$ is the Beltrami differential parameter formed with respect to the form (5), and is defined by

$$\Delta_2\theta = \frac{1}{\sqrt{a}}\sum_i \frac{\partial}{\partial x_i}\left(\sum_k a^{(ik)}\sqrt{a}\frac{\partial\theta}{\partial x_k}\right), \tag{6}$$

where a is the determinant of the functions a_{ik} and $a^{(ik)}$ is the cofactor of a in this determinant divided by a.§ This assumption is equivalent to the equation $G_{44} = 0$. In this equation and hereafter $\sum_j$ means the sum for $j = 1, 2, 3$.

III. The surfaces V = const. form part of a triply orthogonal system in the space, S_3, of coördinates x_1, x_2, x_3.

IV. The orthogonal trajectories of V = const. in S_3 are paths of the particle, in the sense that the coördinates x_1, x_2, x_3, of a world-line determine a path in S_3 of a particle in the gravitational field for which the world-line is the representation in terms of space and time t.

V. The form (5) is euclidean to a first approximation.

2. **Geodesics in the four-space and in S_3.** It can be shown‖ that in any three-space there exist triply-orthogonal systems of surfaces, and accord-

* Rendiconti dei Lincei, ser. 5, vol. 27 (1918), p. 365.

† Ann. der Physik, vol. 54 (1917), p. 117.

‡ Ann. der Physik, vol. 56 (1918), p. 401.

§ Bianchi, Lezioni di Geometrica Differenziale, 2d ed., vol. 1, p. 68.

‖ Wright, Invariants of Quadratic Differential Forms, Cambridge Tract No. 9, pp. 64–67.

ingly (5) can be given the form

$$ds_0^2 = \sum_i a_i dx_i^2, \tag{7}$$

where now the coördinate surfaces form a triply orthogonal system.

If we take s for the parameter along a world-line, and put $\dot{x}_i = dx_i/ds$, $\dot{t} = dt/ds$, the integral (2) becomes

$$\int \sqrt{V^2\dot{t}^2 - \Sigma a_i \dot{x}_i^2}\, ds \equiv \int \varphi(V, x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{t})ds. \tag{8}$$

The Euler equations of condition that (8) be stationary are

$$\frac{\partial \varphi}{\partial x_i} - \frac{d}{ds}\left(\frac{\partial \varphi}{\partial \dot{x}_i}\right) = 0^* \qquad (i = 1, 2, 3, 4;\ x_4 = t). \tag{9}$$

Applying these conditions to (8), and noting that in consequence of the choice of s as parameter, we have $\varphi = 1$ along a world-line, we obtain

$$\frac{d^2x_i}{ds^2} + \sum_j \frac{\partial \log a_i}{\partial x_j}\frac{dx_i}{ds}\frac{dx_j}{ds} - \frac{1}{2a_i}\sum_j \frac{\partial a_j}{\partial x_i}\left(\frac{dx_j}{ds}\right)^2 + \frac{V}{a_i}\frac{\partial V}{\partial x_i}\left(\frac{dt}{ds}\right)^2 = 0, \tag{10}$$

$$\frac{dt}{ds} = \frac{k^2}{V^2}, \tag{11}$$

where k is a constant.

By definition the geodesics is S_3 are the curves along which the integral $\int \sqrt{\Sigma a_i dx_i^2}$ is stationary. When s_0 is taken for the parameter along such a geodesic, we find that the equations of a geodesic are

$$\frac{d^2x_i}{ds_0^2} + \sum_j \frac{\partial \log a_i}{\partial x_j}\frac{dx_i}{ds_0}\frac{dx_j}{ds_0} - \frac{1}{2a_i}\sum_j \frac{\partial a_j}{\partial x_i}\left(\frac{dx_j}{ds_0}\right)^2 = 0. \tag{12}$$

From (4) and (11) it follows that the parameters s and s_0 along a world-line and the corresponding path in S_3 are in the relation

$$ds_0 = \sqrt{\frac{k^2}{V^2} - 1}\, ds. \tag{13}$$

When we express equations (10) in terms of s_0, we obtain

$$\begin{aligned}\frac{d^2x_i}{ds_0^2} + \sum_j \frac{\partial \log a_i}{\partial x_j}\frac{dx_i}{ds_0}\frac{dx_j}{ds_0} - \frac{1}{2a_i}\sum_j \frac{\partial a_j}{\partial x_i}\left(\frac{dx_j}{ds_0}\right)^2 \\ = \frac{k^2}{V(k^2 - V^2)}\left(\frac{dV}{ds_0}\frac{dx_i}{ds_0} - \frac{1}{a_i}\frac{\partial V}{\partial x_i}\right).\end{aligned} \tag{14}$$

From this equation and (12) it follows that a necessary condition that the path of a particle be a geodesic in S_3 is

* Bolza, Lectures on the Calculus of Variations, p. 123.

$$\frac{dV}{ds_0}\frac{dx_i}{ds_0} = \frac{1}{a_i}\frac{\partial V}{\partial x_i} \qquad (i = 1, 2, 3). \tag{15}$$

If we multiply these respective equations by $\sqrt{a_i}$, square the resulting equations and add them, we get

$$\left(\frac{dV}{ds_0}\right)^2 = \sum_i \frac{1}{a_i}\left(\frac{\partial V}{\partial x_i}\right)^2 \equiv \Delta_1 V, \tag{16}$$

where $\Delta_1\theta$ is the first differential parameter of θ with respect to the form (7). When ds_0^2 is written in the general form (5), the expression for $\Delta_1\theta$ is

$$\Delta_1\theta = \sum_{ik} a^{(ik)} \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_k}.^* \tag{17}$$

Hence (15) may be written in the form

$$\frac{dx_i}{ds_0} = \frac{1}{\sqrt{\Delta_1 V}} \cdot \frac{1}{a_i}\frac{\partial V}{\partial x_i}. \tag{18}$$

The direction of the tangent to any curve on a surface V = const. through the point P where a path curve meets the surface is given by the values of dx_i/ds_1, s_1 being the arc along the curve, where

$$\sum_i \frac{\partial V}{\partial x_i}\frac{dx_i}{ds_1} = 0. \tag{19}$$

From (18) and (19) it follows that

$$\Sigma a_i \frac{dx_i}{ds_0}\frac{dx_i}{ds_1} = 0, \tag{20}$$

which is the condition that the path is orthogonal to the surface, since this condition is satisfied by every curve through P.†

Conversely, any orthogonal trajectory of the surfaces V = const. is defined by (18). In fact from (20) and the equation

$$\Sigma a_i \frac{dx_i}{ds_0}\frac{dx_i}{ds_2} = 0,$$

for a second curve on V = const. we get

$$\frac{\dfrac{dx_1}{ds_0}}{a_2a_3\left(\dfrac{dx_2}{ds_1}\dfrac{dx_3}{ds_2} - \dfrac{dx_3}{ds_1}\dfrac{dx_2}{ds_2}\right)} = \cdots = \frac{\dfrac{dx_3}{ds_0}}{a_1a_2\left(\dfrac{dx_1}{ds_1}\dfrac{dx_2}{ds_2} - \dfrac{dx_2}{ds_1}\dfrac{dx_1}{ds_2}\right)} = R, \tag{21}$$

* Bianchi, l. c., p. 61.

† Bianchi, l. c., p. 330.

where by composition we find

$$R = \frac{1}{\sqrt{a_1 a_2 a_3}\,Q}, \qquad Q = \sqrt{\Sigma a_2 a_3 \left(\frac{dx_2}{ds_1}\frac{dx_3}{ds_2} - \frac{dx_3}{ds_1}\frac{dx_2}{ds_2} \right)^2}.$$

In like manner from (19) and

$$\Sigma \frac{\partial V}{\partial x_i}\frac{dx_i}{ds_2} = 0,$$

we obtain

$$(22) \qquad \frac{\dfrac{\partial V}{\partial x_1}}{\dfrac{dx_2}{ds_1}\dfrac{dx_3}{ds_2} - \dfrac{dx_3}{ds_1}\dfrac{dx_2}{ds_2}} = \cdots = \frac{\dfrac{\partial V}{\partial x_3}}{\dfrac{dx_1}{ds_1}\dfrac{dx_2}{ds_2} - \dfrac{dx_2}{ds_1}\dfrac{dx_1}{ds_2}} = \frac{\sqrt{a_1 a_2 a_3}\sqrt{\Delta_1 V}}{Q}.$$

From (21) and (22) follows (18).

The above results may be stated as follows:

If the path of a particle in a permanent gravitational field is a geodesic, it is an orthogonal trajectory of the surfaces $V = \text{const.}$

3. **Condition that the orthogonal trajectories of the surfaces $V = \text{const.}$ be geodesics.** In establishing this condition we make use of the mixed differential parameter of the first order, $\Delta_1(\theta, \varphi)$, which when formed with respect to (5) is defined by

$$\Delta_1(\theta, \varphi) = \sum_{i,k} a^{(ik)} \frac{\partial \theta}{\partial x_i}\frac{\partial \varphi}{\partial x_k}.*$$

For the form (7) this is

$$\Delta_1(\theta, \varphi) = \sum_i \frac{1}{a_i}\frac{\partial \theta}{\partial x_i}\frac{\partial \varphi}{\partial x_i}.$$

From (18) we have

$$\frac{d^2 x_i}{ds_0^2} = \sum_j \frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{\Delta_1 V}}\frac{1}{a_i}\frac{\partial V}{\partial x_i} \right)\cdot \frac{1}{\sqrt{\Delta_1 V}}\frac{1}{a_j}\frac{\partial V}{\partial x_j} = \frac{1}{\sqrt{\Delta_1 V}}\Delta_1\left(V, \frac{1}{\sqrt{\Delta_1 V}}\frac{1}{a_i}\frac{\partial V}{\partial x_i} \right).$$

Substituting in (12), we get

$$a_i \sqrt{\Delta_1 V}\,\Delta_1\left(V, \frac{1}{\sqrt{\Delta_1 V}}\frac{1}{a_i}\frac{\partial V}{\partial x_i} \right) \sum_j \frac{1}{a_j}\frac{\partial \log a_i}{\partial x_j}\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j} - \frac{1}{2}\sum_j \frac{1}{a_j}\frac{\partial \log a_j}{\partial x_i}\left(\frac{\partial V}{\partial x_j} \right)^2 = 0,$$

which may be written

$$(23) \qquad \Delta_1\left(V, \frac{\partial V}{\partial x_i} \right) - \frac{1}{2}\frac{\partial V}{\partial x_i}\frac{\Delta_1(V, \Delta_1 V)}{\Delta_1 V} - \frac{1}{2}\sum_j \frac{1}{a_j}\frac{\partial \log a_j}{\partial x_i}\left(\frac{\partial V}{\partial x_j} \right)^2 = 0,$$

since

* Bianchi, Lezioni, p. 61.

$$(24)\quad \Delta_1(V, \Delta_1 V) = \sum_j \frac{2}{a_j}\,\Delta_1\left(V, \frac{\partial V}{\partial x_j}\right)\frac{\partial V}{\partial x_j} + \sum_j \left(\frac{\partial V}{\partial x_j}\right)^2 \Delta_1\left(V, \frac{1}{a_j}\right).$$

Since

$$\frac{1}{2}\frac{\partial}{\partial x_i}\,\Delta_1 V = \Delta_1\left(V, \frac{\partial V}{\partial x_i}\right) - \frac{1}{2}\sum_j \frac{1}{a_j}\frac{\partial \log a_j}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right)^2,$$

equation (23) may be written

$$\frac{1}{2}\frac{\partial}{\partial x_i}(\Delta_1 V)^2 = \Delta_1(V, \Delta_1 V)\frac{\partial V}{\partial x_i}.$$

Consequently $\Delta_1(V, \Delta_1 V)$, and also $\Delta_1 V$, must be functions of V. But when $\Delta_1 V$ is a function of V so also is $\Delta_1(V, \Delta_1 V)$. Hence we have the theorem:

A necessary and sufficient condition that the orthogonal trajectories of the surfaces V = const. be geodesics is that ΔV be a function of V.

In this case the surfaces V = const. are said to form a *geodesically parallel* family.*

4. **The path of a ray of light.** The function V is interpreted as the velocity of light in the field, and consequently along a world-line of a ray of light $ds = 0$, as follows from (4). In order to obtain the equations of these world-lines, we apply the Fermat principle that $\int dt$ be stationary along such a line, that is the integral $\int \sqrt{V^{-2}\Sigma a_i dx_i^2}$. This gives the equations

$$\frac{d^2x_i}{dt^2} + \frac{dx_i}{dt}\sum_j \frac{\partial}{\partial x_j}\log\frac{a_i}{V^2}\frac{dx_j}{dt} - \frac{V^2}{2a_i}\sum_j \frac{\partial}{\partial x_i}\frac{a_j}{V^2}\left(\frac{dx_j}{dt}\right)^2 = 0.$$

When we require that a path of light be a geodesic in S_3, we obtain (18). Hence:

When the orthogonal trajectories of the surfaces V = const. are paths of a particle in a permanent gravitational field, they are also the paths of a ray of light, and conversely.

5. **Certain triply orthogonal systems in euclidean space.** A necessary and sufficient condition that

$$ds^2 = H_1^2 dx_1^2 + H_2^2 dx_2^2 + H_3^2 dx_3^2$$

the linear element of euclidean space is that the functions H_i satisfy the following equations of Lame:†

* For other proofs of this theorem the reader is referred to Bianchi, l. c., p. 338; also, Wright, l. c., p. 64.

† Eisenhart, Differential Geometry, p. 449.

$$
(25)\qquad
\begin{aligned}
&\frac{\partial^2 H_i}{\partial x_j \partial x_k} = \frac{1}{H_j}\frac{\partial H_j}{\partial x_k}\frac{\partial H_i}{\partial x_j} + \frac{1}{H_k}\frac{\partial H_k}{\partial x_j}\frac{\partial H_i}{\partial x_k},\\
&\frac{\partial}{\partial x_i}\left(\frac{1}{H_i}\frac{\partial H_j}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{H_j}\frac{\partial H_i}{\partial x_j}\right) + \frac{1}{H_k^2}\frac{\partial H_i}{\partial x_k}\frac{\partial H_j}{\partial x_k} = 0,
\end{aligned}
$$

where $i \neq j,\ i \neq k,\ j \neq k$.

We consider the case where H_1 is a function of x_1 alone, and write

$$
(26)\qquad H_1 = X_1',
$$

the accent indicating differentiation with respect to x_1. Now equations (25) reduce to

$$
(27)\qquad \frac{\partial}{\partial x_1}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial x_3}\right) = 0, \qquad \frac{\partial}{\partial x_1}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial x_2}\right) = 0,
$$

$$
(28)\qquad \frac{\partial}{\partial x_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial x_1}\right) = 0, \qquad \frac{\partial}{\partial x_1}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial x_1}\right) = 0,
$$

$$
(29)\qquad \frac{\partial}{\partial x_2}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial x_3}\right) + \frac{1}{H_1^2}\frac{\partial H_2}{\partial x_1}\frac{\partial H_3}{\partial x_1} = 0.
$$

From (28) we have by integration

$$
(30)\qquad H_2 = \sigma X_1 + \bar{\sigma}, \qquad H_3 = \tau X_1 + \bar{\tau},
$$

where σ, $\bar{\sigma}$, τ and $\bar{\tau}$ are independent of x_1. In accordance with (27) these functions must satisfy the conditions

$$
\tau\frac{\partial \bar{\sigma}}{\partial x_3} - \bar{\tau}\frac{\partial \sigma}{\partial x_3} = 0, \qquad \sigma\frac{\partial \bar{\tau}}{\partial x_2} - \bar{\sigma}\frac{\partial \tau}{\partial x_2} = 0.
$$

If we replace these equations by

$$
(31)\qquad
\begin{aligned}
&\frac{\partial \bar{\sigma}}{\partial x_3} = \lambda\bar{\tau}, \qquad \frac{\partial \sigma}{\partial x_3} = \lambda\tau,\\
&\frac{\partial \bar{\tau}}{\partial x_2} = \mu\bar{\sigma}, \qquad \frac{\partial \tau}{\partial x_2} = \mu\sigma,
\end{aligned}
$$

we have

$$
\frac{1}{H_3}\frac{\partial H_2}{\partial x_3} = \lambda = \frac{1}{\tau}\frac{\partial \sigma}{\partial x_3}, \qquad \frac{1}{H_2}\frac{\partial H_3}{\partial x_2} = \mu = \frac{1}{\sigma}\frac{\partial \tau}{\partial x_2},
$$

so that (29) becomes

$$
(32)\qquad \frac{\partial}{\partial x_2}\left(\frac{1}{\sigma}\frac{\partial \tau}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\tau}\frac{\partial \sigma}{\partial x_3}\right) + \sigma\tau = 0.
$$

Each solution of this equation determines λ and μ, and then $\bar{\sigma}$ and $\bar{\tau}$ follow from (31).

Equation (32) expresses the condition that

$$\sigma^2 dx_2^2 + \tau^2 dx_3^2$$

is the linear element of the unit sphere.*

If it is required further that $H_2 H_3$ be the product of a function of x_1 alone, say φ, and a function ψ independent of x_1, we have, from (30)

$$\frac{\sigma\tau}{\psi} X_1^2 + \frac{\sigma\bar{\tau} + \bar{\sigma}\tau}{\psi} X_1 + \frac{\bar{\sigma}\bar{\tau}}{\psi} = \varphi.$$

From the two equations obtained by differentiating this equation with respect to x_2 and x_3 and (31) we find that $\sigma/\bar{\sigma} = \tau/\bar{\tau} = k$, where k is a constants. Consequently the general solution may be written

$$H_2 = X_1\sigma, \qquad H_3 = X_1\tau. \tag{33}$$

6. **Derivation of the Schwarzschild form (3).** In accordance with postulate III, we take the surfaces V = const. for the coördinate surfaces x_1 = const. of a triply orthogonal system in S_3, and write

$$V^2 = c^2(1 + 2\varphi_1(x_1)), \tag{34}$$

where c is the constant velocity of light.

From the results of §§ 2, 3 it follows that postulate IV is equivalent to $\Delta_1 V = \varphi(x_1)$, which reduces to

$$a_1 = \frac{V'^2}{\varphi} = \frac{c^2\varphi_1'^2}{\varphi(1 + 2\varphi_1)}. \tag{35}$$

Since by postulate II we have $\Delta_2 V = 0$, we must have also

$$a_2 a_3 = \frac{\psi^2}{\varphi}, \tag{36}$$

where ψ is independent of x_1.

From the results of § 5 it follows that the linear element of euclidean space, satisfying the conditions that a_1 is a function of x_1 alone and (36), can be given the form

$$ds^2 = dx_1^2 + x_1^2(\sigma^2 dx_2^2 + \tau^2 dx_3^2). \tag{37}$$

In accordance with postulate V the linear element of S_3 is to be (37) to a first approximation. From (35) and (36) we find that such an approximation is given by taking

$$\varphi_1'^2 = \frac{\varphi}{c^2}, \qquad \varphi = \frac{m^2}{4x_1^4}, \tag{38}$$

* Eisenhart, Differential Geometry, p. 157.

where m is a constant, that is $\varphi_1 = -m/2cx_1$. Then from (34), (35) and (37) we have

$$(39) \qquad ds^2 = c^2\left(1 - \frac{m}{cx_1}\right)dt^2 - \frac{1}{1 - \frac{m}{cx_1}}dx_1^2 - x_1^2(\sigma^2 dx_2^2 + \tau^2 dx_3^2).$$

When we take the solutions $\sigma = 1$, $\tau = \sin x_2$ of (32), we obtain (3).

7. Derivation of a form due to Levi-Civita. If in (26) and (33) we put $X_1 = 1/\sqrt{K_0\mu}\,x_1$, where K_0 and μ are constants, we have in place of (37)

$$ds^2 = \frac{1}{K_0\mu x_1^4}dx_1^2 + \frac{1}{x_1^2}\left(\frac{\sigma^2 dx_2^2 + \tau^2 dx_3^2}{K_0\mu}\right),$$

and the expression in parenthesis has curvature $K_0\mu$. In place of (34) we put

$$V^2 = \mu V_0^2(1 + 2\varphi_1(x_1)),$$

where V_0 is a constant. Proceeding as in the preceding section we have in place of (38)

$$\frac{V_0^2\varphi_1'^2}{\varphi} = \frac{1}{K_0 x_1^4\mu^2}, \qquad \varphi = m x_1^4,$$

where m is a constant. If we take accordingly

$$2\varphi_1 = -\frac{\epsilon x_1}{\mu}, \qquad m = \frac{V_0^2 K_0}{4}, \qquad \epsilon = \pm 1,$$

we obtain the form

$$ds^2 = V_0^2(\mu - \epsilon x_1)dt^2 - \frac{dx_1^2}{K_0 x_1^4(\mu - \epsilon x_1)} - \frac{1}{x_1^2}\left(\frac{\sigma^2 dx_2^2 + \tau^2 dx_3^2}{K_0\mu}\right),$$

due to Levi-Civita.*

* L. c.

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ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH A FINITE NUMBER OF EXCEPTIONAL INFINITESIMAL TRANSFORMATIONS.

By S. D. Zeldin.

1. In a former paper* I have discussed the conditions to be imposed on a finite continuous group with a single exceptional infinitesimal transformation in order that some of the structural constants may be simplified. It is the object of this paper to extend these discussions to groups with any finite number of exceptional infinitesimal transformations.

2. Let the group G be generated by $r + g$ (r, g integers) infinitesimal transformations, and let their corresponding operators be $X_1, X_2, \cdots, X_r, X_{r+1}, \cdots, X_{r+g}$, where

$$X_i = \xi_{i,1}(x_1, \cdots, x_{r+g}) \frac{\partial}{\partial x_1} + \cdots \xi_{i,r+g}(x_1, \cdots, x_{r+g}) \frac{\partial}{\partial x_{r+g}}, \qquad (i = 1, 2, \cdots, r+g).$$

These operators must satisfy the symbolic equation

$$(X_i, X_j) = \sum_{k=1}^{r+g} c_{ijk} X_k \dagger \quad (i, j = 1, 2, \cdots, r+g),$$

where $(X_i, X_j) \equiv X_i X_j - X_j X_i$, and the c_{ijk}'s are the structural constants of the group G. Denoting the operators, corresponding to the transformations of the adjoint of G, by the symbols $E_1, E_2, \cdots, E_r, E_{r+1}, \cdots, E_{r+g}$, where

$$E_i \equiv \sum_{j=1}^{r+g} \sum_{k=1}^{r+g} \alpha_j c_{ijk} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r+g),$$

we must have‡

$$(E_i, E_j) = \sum_{k=1}^{r+g} c_{jik} E_k \qquad (i, j = 1, \cdots, r+g).$$

We shall make the following assumptions about the group G:

(a) It has $g(\leq r)$ exceptional infinitesimal transformations.

(b) The adjoint of G' (which, we shall show, is isomorphic with G) has one invariant spread.

* "On the Structure of Finite Continuous Groups with a Single Exceptional Infinitesimal Transformation," Dissertation, Clark University, 1917.

† S. Lie, Continuirliche Gruppen, p. 385.

‡ S. Lie, Continuirliche Gruppen, p. 467.

3. For simplicity let us take $X_{r+1}, X_{r+2}, \cdots, X_{r+g}$ to be the g exceptional infinitesimal transformations of G, i.e.,

$$(X_i, X_j) = \sum_{k=1}^{r+g} c_{ijk} X_k \quad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix}$$

and

$$(X_i, X_j) = 0 \quad \begin{pmatrix} i = 1, \cdots, r+g \\ j = r+1, \cdots, r+g \end{pmatrix}.$$

There will then be a linear relationship with constant coefficients between the operators $E_1, \cdots, E_{r+g}$, i.e., we can find constants $\lambda_1, \cdots, \lambda_{r+g}$, not all zero, for which*

$$(A) \qquad \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_{r+g} E_{r+g} = 0.$$

The solution of equation (A) is evidently $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_r = 0$, while $\lambda_{r+1}, \lambda_{r+2}, \cdots, \lambda_{r+g}$ are arbitrary constants. Since

$$E_{r+1} = E_{r+2} = \cdots = E_{r+g} = 0,$$

it follows that

$$(E_i, E_j) = \sum_{k=1}^{r} c_{ijk} E_k \qquad (i, j = 1, 2, \cdots, r).$$

Therefore, if G contains just g exceptional transformations, there exists a group, say G', with r essential parameters generated by r infinitesimal transformations $Y_1, \cdots, Y_r$, where

$$Y_i \equiv \sum_{k=1}^{r} \xi_{ik}(y_1 \cdots y_r) \frac{\partial}{\partial y_k},$$

such that

$$(Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k \qquad (i, j = 1, 2, \cdots, r).$$

We shall denote the operators of the adjoint of G' by the symbols $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_r$, where

$$\mathcal{E}_i = \sum_{j=1}^{r} \sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k}.\dagger$$

4. Since we assumed that the adjoint of G' has just one invariant, it follows that the nullity of the matrix

$$\sum_{i=1}^{r} \alpha_i \mathcal{E}_i \equiv \begin{vmatrix} \Sigma\alpha_i c_{i11}, & \Sigma\alpha_i c_{i21}, & \cdots, & \Sigma\alpha_i c_{ir1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma\alpha_i c_{i1r}, & & \cdots, & \Sigma\alpha_i c_{irr} \end{vmatrix}$$

* S. Lie, Continuirliche Gruppen, p. 465.

† The α's are the parameters of the group G.

is equal to one,* i.e., at least one of the minors of the determinant $\left|\sum_{i=1}^{r} \alpha_i \mathcal{E}_i\right|$ of order $r - 1$ is not zero. But every minor of $\left|\sum_{i=1}^{r} \alpha_i \mathcal{E}_i\right|$ is also a minor of

$$\left|\sum_{i=1}^{r+g} \alpha_i E_i\right| \equiv \begin{vmatrix} \Sigma\alpha_i c_{i11}, & \cdots, & \Sigma\alpha_i c_{i,r,1}, & 0\cdots 0 \\ \vdots & & \vdots & \\ \Sigma\alpha_i c_{i,1,r}, & \cdots, & \Sigma\alpha_i c_{i,r,r}, & 0\cdots 0 \\ \Sigma\alpha_i c_{i,1,r+1}, & \cdots, & \Sigma\alpha_i c_{i,r,r+1}, & 0\cdots 0 \\ \vdots & & \vdots & \\ \Sigma\alpha_i c_{i,1,r+g}, & \cdots, & \Sigma\alpha_i c_{i,r,r+g}, & 0\cdots 0 \end{vmatrix}.$$

Therefore at least one minor of $\left|\sum_{i=1}^{r+g} \alpha_i E_i\right|$ of order $r - 1$ is not zero, and thus the nullity of the matrix $\Sigma\alpha_i E_i$ can not exceed $g + 1$. Furthermore, for $\alpha_1, \cdots, \alpha_r, \alpha_{r+1}, \cdots, \alpha_{r+g}$ assigned, the symbolic equations

$$(B) \qquad \left(\sum_{i=1}^{r+g} \alpha_i X_i, \sum_{i=1}^{r+g} \eta_i X_i\right) = 0$$

are clearly satisfied for

(1) $\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(2) $\eta_1 = \eta_2 = \cdots = \eta_r = 0, \eta_{r+1} = 1, \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(3) $\eta_1 = \eta_2 = \cdots = \eta_r = \eta_{r+1} = 0, \eta_{r+2} = 1, \eta_{r+3} = \cdots = \eta_{r+g} = 0$

. . . .

($g + 1$) $\eta_1 = \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot = \eta_{r+g-1} = 0, \eta_{r+g} = 1$

But from the equations (B) follows the symbolic system of equations

$$\Sigma\alpha_i E_i(\eta_1, \cdots, \eta_r, \cdots, \eta_{r+g})$$

$$\equiv \begin{pmatrix} \Sigma\alpha_i c_{i1}, & \cdots, & \Sigma\alpha_i c_{ir1}, & 0\cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma\alpha_i c_{i1r+g}, & \cdots, & \Sigma\alpha_i c_{irr+g}, & 0\cdots 0 \end{pmatrix} (\eta_1, \cdots, \eta_r, \cdots, \eta_{r+g}) = 0.\dagger$$

This system has $g + 1$ independent solutions, namely

(1) $\eta_1 = \alpha_1, \eta_2 = \alpha_2, \cdots, \eta_r = \alpha_r, \eta_{r+1} = \cdots = \eta_{r+g} = 0$

(2) $\eta_1 = \cdots = \eta_r = 0, \eta_{r+1} = 1, \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(3) $\eta_1 = \cdots \qquad = \eta_{r+1} = 0, \eta_{r+2} = 1, \eta_{r+3} = \cdots = \eta_{r+g} = 0$

. . . .

. . . .

($g + 1$) $\eta_1 = \cdots = \eta_r = \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot = \eta_{r+g-1} = 0, \eta_{r+g} = 1.$

* See proof in my dissertation, Clark University, 1917.

† This notation is taken from A. Cayley; see Philosophic Transactions, v. 148, 1858, pp. 39–46.

Therefore the nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is at least $g + 1$. The nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is thus just equal to $g + 1$.

5. **The number of invariants of the adjoint of** G. Since the nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is equal to $g + 1$, the adjoint of G must have $g + 1$ independent invariants, i.e., the system of r partial differential equations

$$(C)\quad \begin{aligned} E_1 f(\alpha_1 \cdots \alpha_{r+g}) &\equiv \sum_{j=1}^{r+g} \alpha_j c_{1j1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r+g} \alpha_j c_{1j,\, r+g} \frac{\partial f}{\partial \alpha_{r+g}} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ E_r f(\alpha_1 \cdots \alpha_{r+g}) &\equiv \sum_{j=1}^{r+g} \alpha_j c_{rj1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r+g} \alpha_j c_{1j,\, r+g} \frac{\partial f}{\partial \alpha_{r+g}} = 0, \end{aligned}$$

will be satisfied by $g + 1$ independent functions

$$f = \varphi_1(\alpha_1 \cdots \alpha_{r+g}),\ \varphi_2(\alpha_1 \cdots \alpha_{r+g}),\ \cdots,\ \varphi_{g+1}(\alpha_1 \cdots \alpha_{r+g}).$$

We have assumed that the adjoint of G' has one invariant, i.e., the system of partial differential equations

$$(D)\quad \begin{aligned} \mathcal{E}_1 f(\alpha_1 \cdots \alpha_r) &\equiv \sum_{j=1}^{r} \alpha_j c_{1j1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{1jr} \frac{\partial f}{\partial \alpha_r} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathcal{E}_r f(\alpha_1 \cdots \alpha_r) &\equiv \sum_{j=1}^{r} \alpha_j c_{rj1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{rjr} \frac{\partial f}{\partial \alpha_r} = 0, \end{aligned}$$

is satisfied by only one function $f(\alpha_1 \cdots \alpha_r)$, and all the other solutions of (D) can be expressed as linear functions of f. It is clear that $f(\alpha_1 \cdots \alpha_r)$ (the solution of (D)) will also satisfy the system (C), for,

$$\frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+1}} = \frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+2}} = \cdots = \frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+g}} = 0.$$

We may state, therefore, the following

THEOREM: *If the adjoint of G' has one invariant, the adjoint of G has then $g + 1$ independent invariants, one of which is the invariant of the adjoint of G'.*

6. **The intersections of the spreads invariant to the adjoint of** G. Suppose that $\varphi_1(\alpha)$ is the function, invariant to both the adjoint of G and the adjoint of G'. Let then $\varphi_2(\alpha) = 0$, $\varphi_3(\alpha) = 0$, $\cdots$, $\varphi_{g+1}(\alpha) = 0$, the remaining g spreads invariant to the adjoint of G, be all flats of order $r + g - 2$ in the $r + g - 1$ space, and let their common intersection

(if there is any) be an $r-1$ flat. This flat will be represented by the equations

$$(F)\quad \begin{aligned} \varphi_2(\alpha_1\cdots\alpha_{r+g}) &= 0,\\ \varphi_3(\alpha_1\cdots\alpha_{r+g}) &= 0,\\ \cdot\quad\cdot\quad\cdot & \quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot & \quad\cdot\quad\cdot\\ \varphi_{g+1}(\alpha_1\cdots\alpha_{r+g}) &= 0. \end{aligned}$$

It is evident that this common intersection will also be invariant to the adjoint of G.

Let us consider g points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(i)}, \cdots, \alpha^{(g)}$ in the space of the adjoint of G, the coördinates of $\alpha^{(i)}$ $(i = 1, 2, \cdots, g)$ being all zero except the $(r+i)$th which shall be taken equal to unity. We may take $\alpha^{(i)}$ $(i = 1, 2, \cdots, g)$ to correspond to the invariant subgroups, X_{r+i} $(i = 1, 2, \cdots, g)$ of order one of the group G. Then, in the space of the adjoint of G there will be g points invariant to all the transformations of the adjoint of G.*

If, now, the $(r-1)$ flat (F) does not pass through any of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$, then, by Lie's theorem,† we can take $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r, \overline{X}_{r+1}, \cdots, \overline{X}_{r+g}$ such linear functions of the operators $X_1, X_2, \cdots, X_r, X_{r+1}, \cdots, X_{r+g}$ that

$$\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r$$

will form an invariant subgroup of order r of the adjoint of G, i.e.,

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix}.$$

$$\therefore\ \bar{c}_{ijr+1} = \bar{c}_{ijr+2} = \cdots = \bar{c}_{ijr+g} = 0.$$

7. Let us now suppose that $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$ are not flats but algebraic spreads of orders $m_2, m_3, \cdots, m_{g+1}$ respectively. There will then be g^2 polar flats of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$ taken with respect to $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$. If the spreads $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$ do not pass through any of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$, then the $r-1$ flat, the common intersection (if there is any) of g of the g^2 polar flats, will not pass through any of those points. It will therefore be possible, as we have shown above, to take $\overline{X}_1, \cdots, \overline{X}_{r+g}$ such linear functions of $X_1, \cdots, X_{r+g}$ that $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r$ will form an invariant subgroup of G, i.e.,

$$\bar{c}_{ijr+1} = \bar{c}_{ijr+2} = \cdots = \bar{c}_{ijr+g} = 0.$$

* See S. Lie, Continuirliche Gruppen, p. 468.

† S. Lie, Continuirliche Gruppen, p. 479.

8. The case might arise when $\varphi_2(\alpha)$ passes through one of the points $\alpha^{(1)}, \cdots, \alpha^{(g)}$, say through $\alpha^{(1)}$, but not through the others, and $\varphi_3(\alpha)$ passes through $\alpha^{(2)}$, but not through the others, etc. Then the polar flat of $\alpha^{(1)}$ with respect to $\varphi_2(\alpha)$ will also pass through that point; similarly, the polar flat of $\alpha^{(2)}$ quâ $\varphi_3(\alpha)$ will pass through $\alpha^{(2)}$, etc. The common intersection (if there is any) of those g^2 polar flats may or may not pass through all the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$. If the common intersection is an $r-1$ flat not passing through those points, then we come to the case which we have already discussed above. If, however, it does pass through them, then by Lie's theorem, to which reference was made above, we can choose $\overline{X}_1, \cdots, \overline{X}_{r+g}$ such linear functions of $X_1, \cdots, X_{r+g}$ that $\overline{X}_1, \cdots, \overline{X}_r$ form an invariant subgroup of order r of G, g of which are the operators $X_{r+1}, X_{r+2}, \cdots, X_{r+g}$.

We shall then have

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix};$$

and assuming, for simplicity, that

$$\overline{X}_{r-g+1} = X_{r+1}, \ \overline{X}_{r-g+2} = X_{r+2}, \ \cdots, \ \overline{X}_r = X_{r+g},$$

we shall have

$$(\overline{X}_i, \overline{X}_j) = \sum_{r-g+1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r \\ j = r-g+1, r-g+2, \cdots, r \end{pmatrix}.$$

But that means that G' which has the same structural constants as the corresponding $r-1$ operators of G, has an invariant subgroup of order g, which is contrary to the assumption that the adjoint of G' has one invariant spread, while an invariant subgroup of order g of G' would necessitate an invariant $g-1$ flat of the adjoint of G.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
March 28, 1920.

* S. Lie, Continuirliche Gruppen, p. 479.

CONFORMAL MAPPING OF A FAMILY OF REAL CONICS UPON ANOTHER.*

BY T. H. GRONWALL.

1. **Introduction.** Let $z = x + yi$ and $w = u + vi$ be two complex variables, and $w(z)$ an analytic function of z; then the relation

$$w = w(z) \tag{1}$$

defines a conformal map of the z-plane upon the w-plane. It is the purpose of the present paper to determine *all* functions $w(z)$ such that there exists a family (containing at least one real parameter) of *real* conics in the z-plane which is transformed by (1) into a family (obviously with an equal number of parameters) of real conics in the w-plane. A summary of the results is given in the last paragraph. The particular case where the conics in the w-plane are straight lines parallel to the real axis has been investigated by Von der Mühll† and Meyer.‡ The fact that x and y appear in (1) in the combination $x + yi$ immediately suggests the use of the isometric coördinates

$$z = x + yi, \qquad \bar{z} = x - yi.$$

We have conversely

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}),$$

and

$$x^2 + y^2 = z\bar{z} = |z|^2.$$

2. **Equations of the straight line and circle in isometric coördinates.** The equation of a straight line through $x = p\ (\geqq 0)$, $y = 0$ and perpendicular to the real axis is $x = p$ or $z + \bar{z} - 2p = 0$; replacing z by $ze^{-\alpha i}$ we obtain the equation of a straight line, the perpendicular on which from the origin is of length p and forms an angle α with the real axis:

$$e^{-\alpha i}z + e^{\alpha i}\bar{z} - 2p = 0. \tag{2}$$

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* Read before the American Mathematical Society,

† K. Von der Mühll, "Ueber die Abbildung von Ebenen auf Ebenen," Journ. f. Math., vol. 69 (1868), pp. 264–285.

‡ Hans Meyer, "Ueber die von geraden Linien und von Kegelschnitten gebildeten Isothermen, sowie ueber einige von speciellen Curven dritter Ordnung gebildeten Isothermen," Diss. Göttingen, 1879 (Zürich, Zürcher und Furrer).

The equation of a circle with center at z_1 and radius r is $|z - z_1| = r$ or $(z - z_1)(\bar{z} - \bar{z}_1) = r^2$ or finally

$$z\bar{z} - \bar{z}_1 z - z_1\bar{z} + z_1\bar{z}_1 - r^2 = 0. \tag{3}$$

Consequently, the equation of a straight line or a circle may be written in the form

$$\bar{z} = \frac{az + b}{cz + d}, \tag{4}$$

where

$$ad - bc = 1. \tag{5}$$

When (4) is to be identical with (2), we find $a = -ke^{-\alpha i}$, $b = 2kp$, $c = 0$, $d = ke^{\alpha i}$, the factor of proportionality being determined by (5), which gives $k^2 = -1$. Consequently

$$\begin{aligned} a &= -\epsilon i e^{-\alpha i}, \qquad b = 2\epsilon p i, \\ c &= 0, \qquad d = \epsilon i e^{\alpha i}, \qquad \epsilon = \pm 1, \end{aligned} \tag{6}$$

and the conditions that these equations shall be satisfied by real values of p and α, i.e., that (4) shall represent a real straight line, are

$$\bar{a} = d, \qquad b + \bar{b} = 0, \qquad c = 0. \tag{7}$$

In the same way it is seen that when (4) represents a real circle with center z_1 and radius r, we must have

$$\bar{a} = d, \qquad b + \bar{b} = 0, \qquad c + \bar{c} = 0, \qquad c \neq 0, \tag{8}$$

$$a = \epsilon i \frac{\bar{z}_1}{r}, \qquad b = \epsilon i\left(r - \frac{z_1\bar{z}_1}{r}\right), \qquad c = \frac{\epsilon i}{r}, \qquad d = -\epsilon i \frac{z_1}{r}, \qquad \epsilon = \pm 1. \tag{9}$$

3. **Equations of the other conics in isometric coördinates.** An ellipse (or hyperbola) with foci at 1 and -1 and major (or transverse) axis $2a$ is defined by

$$|z - 1| \pm |z + 1| = 2a. \tag{10}$$

Transposing and squaring, we obtain

$$(z + 1)(\bar{z} + 1) = 4a^2 + (z - 1)(\bar{z} - 1) - 4a|z - 1|$$

or

$$2a|z - 1| = 2a^2 - z - \bar{z};$$

squaring again and solving the resulting quadratic for $\bar{z}$, we find

$$\bar{z} = (2a^2 - 1)z \pm \sqrt{(4a^4 - 4a^2)(z^2 - 1)},$$

or writing $2a^2 - 1 = k$ and omitting the double sign before the square root,

$$\bar{z} = kz + \sqrt{(k^2 - 1)(z^2 - 1)}. \tag{11}$$

From the definition of k, it follows that $k > -1$, and since $a > 1$ when (10) represents an ellipse, it is seen that (11) represents an ellipse when $k > 1$ but an hyperbola when $1 > k > -1$. When $k = 0$, the hyperbola is equilateral.

Replacing z by

$$\frac{2z - z_1 - z_2}{z_1 - z_2}$$

in (11) we find as the equation of an ellipse or hyperbola with foci at z_1 and z_2

$$(12) \quad \begin{aligned} \bar{z} = k\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}z + \frac{\bar{z}_1 + \bar{z}_2}{2} - \frac{k}{2}\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}(z_1 + z_2) \\ + \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}\sqrt{(k^2 - 1)(z - z_1)(z - z_2)}. \end{aligned}$$

The equation of a parabola with the origin as focus, the real axis as axis and the distance $\frac{1}{2}p$ from focus to directrix is

$$|z| = \tfrac{1}{2}(z + \bar{z} + p)$$

or squaring and solving for $\bar{z}$,

$$(13) \quad \bar{z} = z - p + 2i\sqrt{pz}.$$

Replacing z by $(z - z_1)e^{-\alpha i}$, we find the equation of a parabola with focus at z_1 and its axis forming an angle α with the real axis:

$$(14) \quad \bar{z} = e^{-2\alpha i}z + \bar{z}_1 - z_1e^{-2\alpha i} - pe^{-\alpha i} + 2i\sqrt{pe^{-3\alpha i}(z - z_1)}.$$

From (12) and (14) it is seen that the equation of any conic which is not a circle or a straight line, may be written in the form

$$(15) \quad \bar{z} = az + b + \sqrt{cz^2 + 2dz + e},$$

and that its foci are the zeros of the expression under the radical sign.

4. **Condition that two given one-parameter families of real analytic curves shall correspond in a conformal map.** Let $f(z, t)$ be an analytic function of z and t; for real values of t, the equation

(16)

defines a one-parameter family of analytic curves. At a point $z = z_0$, $t = t_0$ where t_0 is real, $\partial f/\partial z \neq 0$ and $f(z_0, t_0) = \bar{z}_0$, the preceding equation takes the form

$$(17) \quad \bar{z} - \bar{z}_0 = \sum_{m+n>0} (a_{mn} + ib_{mn})(z - z_0)^m(t - t_0)^n,$$

where all a and b are real, and $a_{10} + ib_{10} \neq 0$. By Weierstrass' prepara-

tion theorem, this equation may be solved for $z - z_0$, giving as result a power series in $\bar{z} - \bar{z}_0$ and $t - t_0$, and in order that (16) shall represent a family of real curves, it is necessary and sufficient that the solution of (17) in respect to $z - z_0$ shall be identical with the result of changing all complex quantities into their conjugates in (17), or

$$z - z_0 = \sum_{m+n>0} (a_{mn} - ib_{mn})(\bar{z} - \bar{z}_0)^m (t - t_0)^n.$$

Denoting by f or φ a power series in one or more variables, and by $\bar{f}$ or $\bar{\varphi}$ the result of changing its coefficients only into their conjugates, the necessary and sufficient condition that (16) represent a family of real curves is therefore that the equation

$$z = \bar{f}(\bar{z}, t) \tag{18}$$

shall reduce to an identity in z and t upon substitution of the value of $\bar{z}$ given by (16).

Now let

$$\bar{w} = F(w, t) \tag{19}$$

define a family of real analytic curves in the w-plane, and suppose that the analytic function

$$w = w(z) \tag{20}$$

maps the z-plane upon the w-plane in such a manner that the curves (16) are transformed into the curves (19). A necessary condition is found by differentiating (20):

$$dw = w'dz,$$

where $w' = dw/dz$, and taking conjugates, whence

$$d\bar{w} = \bar{w}' \cdot d\bar{z}.$$

By means of (16), (19) and (20), the last equation becomes

$$\frac{\partial F}{\partial w}\frac{dw}{dz}dz + \frac{\partial F}{\partial t}dt = \bar{w}'\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt\right);$$

equating coefficients of dz and dt on both sides, and eliminating $\bar{w}'$, we obtain

$$\frac{\dfrac{\partial F}{\partial w}\dfrac{dw}{dz}}{\dfrac{\partial F}{\partial t}} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}. \tag{21}$$

This condition is also sufficient, for if (20) is an analytic function independent of t which satisfies (21) the equation $d\bar{w} = \bar{w}'d\bar{z}$ becomes by (16) and (21)

$$d\bar{w} = \frac{\bar{w}' \dfrac{\partial f}{\partial t}}{\dfrac{\partial F}{\partial t}} dF(w, t),$$

so that $\bar{w}$ is a function of $F(w, t)$ alone, or

$$F(w, t) = \varphi(\bar{w}). \tag{22}$$

Taking conjugates, and observing that $\bar{F}(\bar{w}, t) = w$ since (19) defines a family of real curves, we find $w = \bar{\varphi}(w)$, and consequently $\bar{w} = \varphi(\bar{w})$, so that the curves (22) into which (16) are transformed by (20), are identical with the given curves (19). Writing (21) for two different values of t, say t_0 and t, we may eliminate dw/dz and obtain

$$\frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}} : \left(\frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}}\right)_{t=t_0} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}} : \left(\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}\right)_{t=t_0}. \tag{23}$$

If this is an identity in w, z and t, each member must be a function of t alone, say $\psi(t)$, and we have

$$\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}} = \varphi(z)\psi(t), \qquad \frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}} = \Phi(w)\psi(t), \qquad \Phi(w)\frac{dw}{dz} = \varphi(z). \tag{24}$$

In this case, the z- and w-planes may be mapped on an auxiliary Z-plane in such a manner that the curves (16) and (19) correspond to a family of parallel straight lines in the Z-plane. For define the map of the z-plane on the Z-plane by

$$\frac{dZ}{dz} = \varphi(z); \tag{25}$$

the curves (16) are then evidently mapped into a family of real curves $\bar{Z} = G(Z, t)$ in the Z-plane, and we have the equation, corresponding to (21),

$$\frac{\dfrac{\partial G}{\partial Z}}{\dfrac{\partial G}{\partial t}} \frac{dZ}{dz} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}. \tag{26}$$

or using (25) and the first of (24),

$$\frac{1}{\psi(t)} \frac{\partial G}{\partial Z} - \frac{\partial G}{\partial t} = 0. \tag{27}$$

The left member being the jacobian of G and $Z + \int[dt/\psi(t)]$, the latter expression is a function $\chi(G)$ of G, so that the equation of the curves in the Z-plane corresponding to (16) is

$$\chi(\bar{Z}) = Z + \int \frac{dt}{\chi(t)};$$

changing to conjugates, we have

$$\bar{Z} + \int \frac{dt}{\bar{\psi}(t)} = \bar{\chi}(Z)$$

so that, since $\bar{Z} = G(Z, t)$,

$$G(Z, t) = \bar{\chi}(Z) - \int \frac{dt}{\bar{\psi}(t)}.$$

Substituting in (27), we find

$$\bar{\chi}'(Z) = -\frac{\psi(t)}{\bar{\psi}(t)} = \text{constant} = a,$$

and ψ and $\bar{\psi}$ being conjugates, we have $|a| = 1$ or $a = -e^{-2ai}$. The curves $\bar{Z} = G(Z, t)$ are consequently the parallel straight lines

$$e^{-ai}Z + e^{ai}\bar{Z} + \int \frac{e^{-ai}}{\psi(t)}dt = 0.$$

Conversely, it follows from (26) that when the curves $\bar{Z} = G(Z, t)$ are parallel straight lines, the quotient $(\partial f/\partial z) : (\partial f/\partial t)$ must be the product of a function of z by a function of t. We remark finally that when (23) is not an identity in w, z and t, this equation gives w as a function of z without integration, and if (16) and (19) are both algebraic curves, $w(z)$ is an algebraic function of z.

5. **Cases where the conics in both the z- and the w-plane are all circles or straight lines.** When a one-parameter family of circles or straight lines in the z-plane

$$\text{(28)} \qquad \bar{z} = \frac{a(t)z + b(t)}{c(t)z + d(t)}, \qquad ad - bc = 1$$

is mapped into the circles or straight lines in the w-plane

$$\text{(29)} \qquad \bar{w} = \frac{A(t)w + B(t)}{C(t)w + D(t)}, \qquad AD - BC = 1,$$

the equation (21) takes the form

$$\text{(30)} \qquad (lz^2 + mz + n)\frac{dw}{dz} = Lw^2 + Mw + N,$$

where

$$
\begin{aligned}
l &= a'(t)c(t) - c'(t)a(t), \\
m &= a'(t)d(t) - d'(t)a(t) + b'(t)c(t) - c'(t)b(t), \qquad (31)\\
n &= b'(t)d(t) - d'(t)b(t)
\end{aligned}
$$

with corresponding expressions for L, M, N in terms of A, B, C, D. Giving t a constant value t_0 in (30), this equation takes the form

$$\frac{dw}{L_0 w^2 + M_0 w + N_0} = \frac{dz}{l_0 z^2 + m_0 z + n_0};$$

using the well-known property of a fractional linear substitution on z to transform any circle into a circle, we may reduce the last equation to a simpler form. Denoting the zeros of $l_0 z^2 + m_0 z + n_0$ by z_1 and z_2 (or the single finite zero by z_1 when $l_0 = 0$), we have

$$
\begin{aligned}
\frac{dz}{l_0 z^2 + m_0 z + n_0} &= \frac{1}{l_0(z_1 - z_2)} \cdot \frac{z - z_2}{z - z_1} d\left(\frac{z - z_1}{z - z_2}\right) && \text{when} \quad z_1 \neq z_2, \\
&= d\left(\frac{1}{l_0(z_1 - z)}\right) && \text{"} \quad z_1 = z_2, \\
&= \frac{1}{m_0} \cdot \frac{1}{z - z_1} d(z - z_1) && \text{"} \quad l_0 = 0,\ m_0 \neq 0, \\
&= d\left(\frac{1}{n_0} z\right) && \text{"} \quad l_0 = m_0 = 0,
\end{aligned}
$$

and making the linear substitution indicated by the form of the expressions to the right, as well as the corresponding substitutions on w, our equation may be reduced to one of the following four forms, k being constant:

$$
\begin{aligned}
\frac{dw}{w} &= k\frac{dz}{z}, \\
dw &= \frac{dz}{z}, \qquad (32)\\
\frac{dw}{w} &= dz, \\
dw &= dz.
\end{aligned}
$$

Consider first the case (32_1). Substituting the value of dw/dz from this equation in (30) the latter becomes

$$Lzw^2 + [-klz^2 + (M - km)z - kn]w + Nz = 0;$$

differentiating, and substituting the value of dw/dz from (32_1), we find

$$(2k + 1)Lzw^2 + [-k(k+2)lz^2 + (k+1)(M - km)z - k^2n]w + Nz = 0,$$

and subtracting and dividing by kw,

$$2Lzw - (k+1)lz^2 + (M - km)z - (k-1)n = 0.$$

Differentiating, substituting the value of dw/dz, and eliminating w, we obtain

$$(k^2 - 1)lz^2 - k(M - km)z + (k^2 - 1)n = 0.$$

When $k = \pm 1$, (32_1) gives $w = \text{const.}\, z$ and $w = \text{const.}/z$ respectively, and these linear substitutions transform every circle into a circle. When $k^2 \neq 1$, we have

$$l = 0, \qquad M = km, \qquad n = 0,$$

the preceding equations give

$$L = 0, \qquad N = 0,$$

and (30) reduces to (32_1).

In discussing the equations $l = n = 0$, we distinguish two cases. First, let $a \neq 0$; then, by (31), $l = 0$ gives $c = c_1 a$, and $n = 0$, $b = c_2 d$, where c_1 and c_2 are constants. By the conditions of reality (7) or (8), we have $d = \bar{a}$ so that $b = c_2\bar{a}$, hence $a =$ constant would make b, c and d constants, and our family of circles would reduce to a single circle. Equation (5) now gives $a\bar{a}(1 - c_1c_2) = 1$, and $c + \bar{c} = 0$, $c_1 a + \bar{c}_1\bar{a} = 0$, or multiplying by a and using the preceding equation, $c_1(1 - c_1c_2)a^2 + c_2 = 0$; since a is not a constant, we must have $c_1 = c_2 = 0$ or $b = c = 0$. From (6) it now follows that

$$a = -\epsilon i e^{-\theta i}, \qquad d = \epsilon i e^{\theta i}, \qquad p = 0,$$

so that (28) represents all straight lines through the origin, and (31_2) gives

$$m = -2i\frac{d\theta}{dt}.$$

The second case, $a = 0$, gives $d = \bar{a} = 0$, and consequently $l = n = 0$. Since now $c \neq 0$ by (5), it follows from (9) that

$$b = \epsilon i r, \qquad c = \frac{\epsilon i}{r}, \qquad z_1 = 0,$$

so that (28) represents all circles with the origin as center, and by (31)

$$m = -2\frac{dr}{dt}.$$

Similarly, the equations $L = N = 0$ give rise to the straight lines through the origin with

$$A = -\epsilon i e^{-\vartheta i}, \qquad D = \epsilon i e^{\vartheta i}, \qquad M = -2i\frac{d\vartheta}{dt},$$

and the circles with the origin as center with

$$B = \epsilon i R, \qquad C = \frac{\epsilon i}{R}, \qquad M = -2\frac{dR}{dt}.$$

Since θ, ϑ, r and R are real, the equation $M = km$ now shows that the constant k is either real or purely imaginary. In the former case, the straight lines through the origin correspond in the two planes, and so do the concentric circles. When k is purely imaginary, the straight lines in one plane correspond to the circles in the other.

Proceeding to the case (32_2), we obtain from (30)

$$Lw^2 + Mw + N - lz - m - \frac{n}{z} = 0.$$

Differentiating, using (32_2) and multiplying by z, we find

$$2Lw + M - lz + \frac{n}{z} = 0,$$

and repeating this process,

$$2L - lz - \frac{n}{z} = 0,$$

whence $l = n = 0$. Hence, as in the preceding case, the curves (28) are the straight lines through the origin and the circles with the origin as center. On a line through the origin, we have $\bar{z}/z =$ constant, and hence, by (32_2),

$$d\bar{w} - dw = \frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} = 0,$$

or $\bar{w} - w =$ constant. Similarly, for a circle with the origin as center, $z\bar{z} =$ const., and by (32_2), $\bar{w} + w =$ const. Consequently, in the case (32_2), $w = \log z +$ const. maps the straight lines through the origin in the z-plane into the lines parallel to the real axis in the w-plane and the circles with the origin as center in the z-plane into the straight lines parallel to the imaginary axis in the w-plane. The case (32_3) reduces to the preceding upon replacing z by w, and (32_4) obviously transforms all circles or straight lines into circles or straight lines.

6. **Cases where the conics in one plane are neither circles nor straight lines.** Permuting w and z if necessary, we may assume that the conics in question lie in the z-plane and have the equation (15) with coefficients depending on t, or

(33) $$\bar{z} = a(t)z + b(t) + \sqrt{c(t)z^2 + 2d(t)z + e(t)}.$$

When the corresponding conics in the w-plane are the straight lines or circles (29), equation (21) becomes

$$(34)\qquad \frac{1}{Lw^2 + Mw + N}\frac{dw}{dz} = \frac{a\sqrt{cz^2 + 2dz + e} + cz + d}{(a'z + b')\sqrt{cz^2 + 2dz + e} + \tfrac{1}{2}(c'z^2 + 2d'z + e')},$$

where $a' = da(t)/dt$ etc., and L, M and N are defined by (31).

When the conics in the w-plane are

$$(35)\qquad \bar{w} = A(t)w + B(t) + \sqrt{C(t)w^2 + 2D(t)w + E(t)},$$

equation (21) becomes

$$(36)\qquad \frac{A\sqrt{Cw^2 + 2Dw + E} + Cw + D}{(A'w + B')\sqrt{Cw^2 + 2Dw + E} + \tfrac{1}{2}(C'w^2 + 2D'w + E')}\frac{dw}{dz} = \frac{a\sqrt{cz^2 + 2dz + e} + cz + d}{(a'z + b')\sqrt{cz^2 + 2dz + e} + \tfrac{1}{2}(c'z^2 + 2d'z + e')}.$$

The foci of the conics in the z-plane being the branch points of the radical in the right-hand members of (34) and (36), we shall distinguish the case when one at least of these foci is variable with t from the case where the foci are fixed, and the former case will be subdivided according as the variable foci are or are not branch points of the right-hand members in (34) and (36).

7. Cases where the conics in the z-plane have a focus variable with t but which is not a branch point of the right-hand member in (34) and (36). When z turns around the focus in question, the radical in (33) changes its sign, and by hypothesis, the right-hand member in (34) or (36) remains unchanged. Being a linear fraction in the radical, this right-hand member must consequently be independent of it, that is

$$\tfrac{1}{2}a(c'z^2 + 2d'z + e') - (a'z + b')(cz + d) = 0,$$

whence

$$(37)\qquad \begin{aligned} ac' - 2a'c &= 0, \\ ad' - a'd - b'c &= 0, \\ ae' - 2b'd &= 0. \end{aligned}$$

First, assume that $a(t)$ is not identically zero. The differential equations (37) are then readily integrated and give

$$(38)\qquad \begin{aligned} c &= c_1a^2, \\ \frac{d}{a} &= c_1b + c_2, \\ e &= c_1b^2 + 2c_2b + c_3, \end{aligned}$$

where c_1, c_2, c_3 are constants. Assuming our conics to be the central conics (12), where k, z_1 and z_2 now depend on t, (38_1) gives

$$\frac{k^2 - 1}{k^2} = c_1,$$

so that k is a constant; (38_2) now becomes

$$\frac{k^2 - 1}{2k} (\bar{z}_1 + \bar{z}_2) + c_2 = 0,$$

so that $z_1 + z_2$ is a constant, which may be made equal to zero by a change of origin. Hence $z_2 = - z_1$, $c_2 = 0$ and $b = 0$, (38_3) gives $(l - k^2)z_1^2 = c_3$, so that z_1 is a constant, and we have a single conic instead of a family. When our conics are the parabolas (14), we have $c = 0$ whence $c_1 = 0$ by (38_1), and (38_2) becomes

$$- 2pe^{-\alpha i} = c_2,$$

so that both p and α are constants. By a rotation of the axes, we may make $\alpha = 0$, and (38_3) becomes $4p(p - \bar{z}_1) + c_3 = 0$, so that z_1 is a constant, and we have again a single conic instead of a family. There remains the case where $a(t) = 0$, and since the coefficient of z in (14) is different from zero, we have only to consider (12) with $k = 0$, representing a family of equilateral hyperbolas. Since $c \neq 0$, (37_2) gives $b' = 0$ or $\bar{z}_1 + \bar{z}_2 =$ const., and (37_3) is satisfied; moving the origin, we may make the constant equal to zero, so that $z_2 = - z_1$, and (12) becomes

(39) $$\bar{z} = \frac{\bar{z}_1}{z_1} \sqrt{z_1^2 - z^2}.$$

Writing

(40) $$z_1 = re^{\theta i},$$

we have

(41) $$c = - e^{-4\theta i}, \qquad d = 0, \qquad e = r^2e^{-2\theta i},$$

and the right-hand member of (34) and (36) being independent of the radical, it is equal to

(42) $$\frac{2cz}{c'z^2 + e'}.$$

Consider first the case where the z-plane may be mapped on an auxiliary Z-plane, the conics in the z-plane corresponding to parallel straight lines in the Z-plane. By § 4, this is the case when (42) becomes the product of a function of z by a function of t, which is possible in two ways: $c' = 0$ or $c' \neq 0$ and e'/c' constant. Assuming $c' = 0$, it follows from (41) that θ is a constant (that is, a parameter independent of t). By (25) and (42),

we obtain $dZ/dz = 2z$, so that making the integration constant zero by moving the origin of the Z-plane

(43) $$Z = z^2,$$

and by (39) and (41), it is seen that the family of all equilateral hyperbolas with the origin as center (having two parameters r and θ)

(44) $$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

is transformed by (43) into the family of all straight lines in the Z-plane

(45) $$e^{-2\theta i}Z + e^{2\theta i}\bar{Z} = r^2.$$

In particular, giving θ a constant value, (44) becomes the family of all equilateral hyperbolas with the origin as center and their foci on the straight line $z_1 = re^{\theta i}$, while (45) represents parallel straight lines. For those lines (45) which pass through a fixed point $Z_0 = a^2e^{2\alpha i}$ we have

$$r^2 = e^{-2\theta i}Z_0 + e^{2\theta i}\bar{Z}_0 = 2a^2 \cos 2(\theta - \alpha),$$

and the corresponding equilateral hyperbolas in the z-plane have their foci at

(46) $$\pm z_1 = \pm a\sqrt{2 \cos 2(\theta - \alpha)} \cdot e^{\theta i}.$$

The equations of the hyperbolas and straight lines in question are

$$e^{-2\theta i}(z^2 - a^2e^{2\alpha i}) + e^{2\theta i}(\bar{z}^2 - a^2e^{-2\alpha i}) = 0,$$

$$e^{-2\theta i}(Z - a^2e^{2\alpha i}) + e^{2\theta i}(\bar{Z} - a^2e^{-2\alpha i}) = 0.$$

Multiplying z and Z by suitable constants, we may make $a = 1$ and $\alpha = 0$, and it now follows from (25) and (42) that the function

(47) $$Z = \log (z^2 - 1)$$

maps the one-parameter family of equilateral hyperbolas

(48) $$e^{-2\theta i}(z^2 - 1) + e^{2\theta i}(\bar{z}^2 - 1) = 0$$

with center at the origin and foci

(49) $$\pm z_1 = \pm \sqrt{2 \cos 2\theta}e^{\theta i}$$

situated at the end points of the diameters of a lemniscate with its foci at ± 1, upon the straight lines

(50) $$\bar{Z} - Z = -\left(\frac{\pi}{2} + 4\theta\right)i$$

parallel to the real axis in the Z-plane.

We now pass on to the cases where $c' \neq 0$ and e'/c' is not independent

of t; from the remark at the end of § 4, it follows that w is an algebraic function of z. Let us assume first that the conics in the w-plane are the straight lines or circles (29); by (42), it follows that (34) becomes

$$\frac{1}{Lw^2 + Mw + N}\frac{dw}{dz} = \frac{2cz}{c'z + e'}, \tag{51}$$

and giving t a constant value, making a linear transformation on w according to (32), and observing that w must be an algebraic function of z, we obtain from (51)

$$\frac{1}{w}\frac{dw}{dz} = \frac{2mz}{z^2 + c_1^2},$$

whence

$$w = (z^2 + c_1^2)^m,$$

where m is real and rational. Replacing, if necessary, w by $1/w$, we may assume $m > 0$, and substituting in (51), we find

$$m(z^2 + c_1^2)^{m-1}(c'z^2 + e') = cL(z^2 + c_1^2)^{2m} + cM(z^2 + c_1^2)^m + cN.$$

Letting $z \to \infty$, we find $L = 0$, $mc' = cM$, so that the preceding equation becomes

$$m(z^2 + c_1^2)^{m-1}(e' - c_1^2c') = cN,$$

and since $e' - c_1^2c'$ is not identically zero by hypothesis, we must have $m = 1$, whence, moving the origin in the w-plane, so that $c_1 = 0$,

$$w = z^2.$$

Since (44) represents all equilateral hyperbolas in the z-plane with the origin as center, it follows at once that the present case reduces to that defined by (43), (44) and (45) upon replacing w by Z. In the second place, assume the conics in the w-plane to be (35); since w is an algebraic function of z, and the right-hand member in (36) is (42) and has no branch points variable with t, the same must be true of the left-hand member, i.e., this must be of the type (42) unless the foci of (35) are fixed. Consequently (36) reduces to

$$\frac{2Cw}{C'w^2 + E'}\frac{dw}{dz} = \frac{2cz}{c'z^2 + e'}.$$

Giving t a constant value and multiplying w by a suitable constant, we obtain a differential equation giving upon integration

$$w^2 + c_2^2 = (z^2 + c_1^2)^m,$$

where m is rational since $w(z)$ is algebraic. Substituting in the preceding equation, we obtain

$$(mCc' - cC')(z^2 + c_1^2)^m + mC(c' - c_1^2c')(z^2 + c_1^2)^{m-1} - c(E' - c_2^2C') = 0,$$

and if $m \neq 1$, it follows that $c' = c_1^2c'$ contrary to hypothesis. Hence $m = 1$, and writing c_1^2 instead of $c_1^2 - c_2^2$, we find

$$w^2 = z^2 + c_1^2 \tag{52}$$

where c_1 may be assumed real (multiplying z and w by a constant).

Noting that (44) represents all the equilateral hyperboles (39) under consideration, it follows at once that (52) transforms the two parameter family of equilateral hyperbolas (44) into itself with a changed parameter r:

$$e^{-2\theta i}w^2 + e^{2\theta i}\bar{w}^2 = r^2 + 2c_1^2 \cos 2\theta. \tag{53}$$

There finally remains the case where the foci of (35) are fixed, and permutating z and w, this falls under the cases treated in § 9 and 10.

8. **Cases where the conics in the z-plane have a focus variable with t which is a branch point of the right-hand member in (34) and (36).** In any such case, the right-hand member in (34) and (36) cannot be of the form $\varphi(z)\psi(t)$ since the branch points of $\varphi(z)$ are fixed. By the remark at the end of § 4, w is therefore an algebraic function of z. Now the branch points of the left-hand member in (34) are those of $w(z)$ and consequently fixed, while the right-hand member has at least one variable branch point by hypothesis. This case is, therefore, excluded and there remains the case (36) where the left-hand member cannot be independent of the radical (in which case all the branch points would be fixed). Hence we may solve (36) for the w-radical and obtain

$$\sqrt{(w - w_1)(w - w_2)} = \varphi(z, t)\sqrt{(z - z_1)(z - z_2)}, \tag{54}$$

where $w_1 = w_1(t)$ and $w_2 = w_2(t)$ are the foci in the w-plane, both supposed variable, and similarly z_1 and z_2 the variable foci in the z-plane; $\varphi(z, t)$ is rational in z, w and dw/dz and is, therefore, an algebraic function with fixed branch points (viz., those of w). If only one focus in the w-plane is variable, we divide by the factor $\sqrt{w - w_2}$ which is independent of t, and obtain

$$\sqrt{w - w_1} = \varphi(z, t)\sqrt{(z - z_1)(z - z_2)}, \tag{54a}$$

where the branch points of the algebraic function $\varphi(z, t)$ are now those of w and eventually also the points where $w = w_2$, that is the branch points of φ are again fixed. There are similar modifications when only one focus in the z-plane is variable. In (54), $\varphi(z, t)$ cannot have a fixed zero except $z = \infty$ since for such a zero z_0 we must have $w(z_0) = w_1(t)$ or $w_2(t)$, while both the latter are variable. Moreover, $\varphi(z, t)$ cannot have a

variable zero $z(t)$ since, excluding those constant t-values for which $z(t)$ coincides with one of the (fixed) branch points of φ, or for which $w_1(t) = w_2(t)$, we have then a holomorphic expansion

$$\varphi(z, t) = \varphi_0(t)(z - z(t)) + \cdots$$

and (54) gives, assuming for instance that $w = w_1(t)$ for $z = z(t)$,

$$w = w_1(t) + \lambda(t)(z - z(t))^m + \cdots,$$

where $m \geqq 2$, and consequently

$$\frac{dw}{dz} = 0 \qquad \text{for} \qquad z = z(t),$$

which is impossible, the zeros of dw/dz being fixed. Similarly, we see that $\varphi(z, t)$ can have no pole variable with t. Hence $z = z_1(t)$ makes the radical to the left in (54) vanish, so that $w = w_1(t)$ or $w_2(t)$. Consequently w takes only two values for each value of z, and in the same way it is seen that z takes only two values for each value of w, so that the algebraic equation connecting w and z is of the second degree in either (in the case (54a), it is of the first degree in w). By (54), φ^2 is therefore a two-valued algebraic function of z and satisfies an equation of the form

$$\alpha(z, t)\varphi^4 + 2\beta(z, t)\varphi^2 + \gamma(z, t) = 0,$$

where α, β and γ are polynomials in z, and since φ has no zero except $z = \infty$ and no pole variable with t, it follows that

$$\gamma(z, t) = \gamma(t), \qquad \alpha(z, t) = \alpha(z)\alpha_1(t),$$

and we find

$$\varphi^2 = \frac{-\beta(z, t) \pm \sqrt{\beta(z, t)^2 - \alpha(z)\alpha_1(t)\gamma(t)}}{\alpha(z)\alpha_1(t)}.$$

But φ^2 has no branch points variable with t, and consequently

$$\beta(z, t)^2 = \alpha(z)\alpha_1(t)\gamma(t) + \delta(z)\delta_1(t)$$

where $\delta(z)$ is a polynomial. If $\delta(z)$ equals a constant times $\alpha(z)$, it is seen at once that φ^2 takes the form

$$\varphi^2 = \frac{1}{\sqrt{\alpha(z)}}\chi(t).$$

Suppose now that $\delta(z)$ and $\alpha(z)$ are linearly independent; any zero of $\beta(z, t)$ which is variable with t must satisfy the equations

$$\alpha(z)\cdot\alpha_1\gamma + \delta(z)\cdot\delta_1 = 0,$$
$$\alpha'(z)\cdot\alpha_1\gamma + \delta'(z)\cdot\delta_1 = 0,$$

and consequently $\alpha(z)\delta'(z) - \alpha'(z)\delta(z) = 0$, so that $\delta(z) = \text{const.}\ \alpha(z)$ contrary to the hypothesis. Consequently the zeros of $\beta(z, t)$ are fixed, so that $\beta(z, t) = \beta(z)\beta_1(t)$, and the identity

$$\beta(z)^2\beta_1(t)^2 = \alpha(z)\alpha_1(t)\gamma_1(t) + \delta(z)\delta_1(t)$$

gives

$$0 = \alpha(z)\cdot\frac{d}{dt}\frac{\alpha_1\gamma}{\beta_1{}^2} + \delta(z)\cdot\frac{d}{dt}\frac{\delta_1}{\beta_1{}^2};$$

the linear independence of $\alpha(z)$ and $\delta(z)$ requires that $\beta_1(t)^2 = \text{const.}\ \alpha_1(t)\gamma(t)$, $\delta_1(t) = \text{const.}\ \alpha_1(t)\gamma(t)$, and φ^2 takes the form

$$\varphi^2 = \sqrt{\frac{\gamma(t)}{\alpha_1(t)}}\cdot\psi(z).$$

It is thus shown that (54) may always be written in the form

$$(w - w_1)(w - w_2) = \psi(z)\chi(t)(z - z_1)(z - z_2) \tag{55}$$

(and similarly 54a). Now differentiate in respect to t:

$$-(w_1 + w_2)'\cdot w + (w_1w_2)' = \psi(z)[\chi'z^2 - (\chi(z_1 + z_2))'z + (\chi z_1z_2)'],$$

and if $(w_1 + w_2)'$ and $(w_1w_2)'$ are linearly independent, we may eliminate $\psi(z)$ by writing this equation for two different values of t and dividing member by member, thus obtaining w as a rational function of z. In this case, writing (55) in the form

$$(z - z_1)(z - z_2) = \frac{1}{\psi(z)}\cdot\frac{1}{\chi(t)}(w - w_1)(w - w_2),$$

and proceeding in the same way, we find that when $(z_1 + z_2)'$ and $(z_1z_2)'$ are linearly independent, z is also a rational function of w, that is, the algebraic equation connecting w and z is of the first degree in either. The degenerate cases like (54a) lead to the same conclusions, and permuting eventually w and z, we see that there are only two cases to be considered: First, the relation between w and z is

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

($\alpha, \beta, \gamma, \delta$ constants) and second, a linear relation exists between $(z_1 + z_2)'$ and $(z_1z_2)'$. In the first case, suppose $\gamma \neq 0$; replacing w and z by expressions of the form $lw + m$, $nz + p$, the relation between them may be reduced to the form

$$w = \frac{1}{z} \tag{56}$$

and our conics in the z-plane, which are not circles or straight lines by hypothesis, are transformed into curves of the fourth degree in the w-plane.*

There remains the assumption $\gamma = 0$, or

$$(57) \qquad w = mz + n,$$

which evidently transforms the five-parameter family of all real conics is the z-plane into the corresponding family in the w-plane.

In the second case, we have either $(z_1 + z_2)' = 0$, so that $z_1 + z_2 = c_1$, or $(z_1z_2)' = c_2(z_1 + z_2)'$, so that $(z_1 - c_2)(z_2 - c_2) = c_3$. By suitable linear transformations, these relations are reduced to the forms $z_1 + z_2 = 0$ and $z_1z_2 = 1$ respectively.

First, let

$$z_1(t)z_2(t) = 1,$$

and denote by

$$G(z, w) = 0$$

the algebraic equation connecting z and w, and which is of the second degree in z. Since $z_1(t)$ and $z_2(t)$ are the two roots of the equation

$$G(z, w_1(t)) = 0,$$

and $w_1(t)$ is variable, it follows that for any w, the two roots of $G(z, w) = 0$ are z and $1/z$. The two equations

$$G(z, w) = 0 \qquad \text{and} \qquad G\left(\frac{1}{z}, w\right) = 0$$

being identical, it follows that both transform the given family of focal conics in the z-plane into the given family of conics in the w-plane, and consequently the relation

$$Z = \frac{1}{z}$$

transforms our family of focal conics in the z-plane into itself, which conclusion we have recognized as impossible in connection with (56).

Now let

$$z_1(t) + z_2(t) = 0,$$

so that $d = 0$, and by (12), $b = 0$. By the same argument as before, it follows that the two roots of $G(z, w) = 0$ are z and $-z$. Moving the origin of the w-plane, we may assume that one of the branch points of the

* This is most readily seen by using rectangular coördinates and transforming an equation of the second degree in x and y by the formulas

$$x = \frac{u}{u^2 + v^2}, \qquad y = \frac{-v}{u^2 + v^2}.$$

algebraic function $z(w)$ is $w = 0$. When w makes a circuit about the origin, z changes into $-z$. Now $e(t)$ is not identically zero, since then $z_1 = z_2 = 0$, and if $E(t)$ is not identically zero, we may choose an $\epsilon > 0$ and an interval for t such that $|e(t)| > \epsilon$, $|E(t)| > \epsilon$. Making the circuit about $w = 0$ sufficiently small, it will not, therefore, enclose any branch point of $\sqrt{Cw^2 + 2Dw + E}$ or of $\sqrt{cz^2 + e}$, and the result of continuing (36) analytically along this circuit is then, remembering that $b = d = 0$,

$$\frac{A\sqrt{Cw^2 + 2Dw + E} + Cw + D}{(A'w + B')\sqrt{Cw^2 + 2Dw + E} + \frac{1}{2}(C'w^2 + 2D'w + E')}\frac{dw}{d(-z)}$$
$$= -\frac{a\sqrt{cz^2 + e} - cz}{-a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')}.$$

Comparing this to (36) we find

$$\frac{a\sqrt{cz^2 + e} + cz}{a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')} = \frac{-a\sqrt{cz^2 + e} + cz}{-a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')},$$

which is exactly the condition, investigated in § 7, that the right-hand member in (36) shall have no variable branch point. This case is therefore to be discarded, and we must assume that $E(t)$ vanishes identically. Then $w_2 = 0$, $G(z, w)$ is of the first degree in w, and $G(z, w) = G(-z, w)$ identically; since $w = 0$ is a branch point of $z(w)$, it follows that $G(z, w) = 0$ takes the form

$$w = \frac{\alpha z^2}{\beta z^2 + \gamma}.$$

According as $\beta \neq 0$ or $\beta = 0$, this reduces, upon multiplying z and w by suitable constants, to

$$w = \frac{z^2}{z^2 + 1} \quad \text{or} \quad w = z^2,$$

and the conics

$$\bar{z} = az + \sqrt{cz^2 + e}$$

and

$$\bar{w} = Aw + B + \sqrt{w(Cw + 2D)}$$

correspond.

In the first case,

$$\bar{w} = A\frac{z^2}{z^2 + 1} + B + \frac{z\sqrt{(C + 2D)z^2 + 2D}}{z^2 + 1},$$

and since we also have

$$\bar{w} = \frac{\bar{z}^2}{\bar{z}^2 + 1},$$

it follows that

$$(58)\qquad \frac{(A+B)z^2+B+z\sqrt{(C+2D)z^2+2D}}{z^2+1} = \frac{(a^2+c)z^2+e+2az\sqrt{cz^2+e}}{(a^2+c)z^2+e+1+2az\sqrt{cz^2+e}}.$$

When $a = 0$, the conics in the z-plane are the equilateral hyperbolas (44) which are mapped by $Z = z^2$ on the straight lines (45), and

$$w = \frac{Z}{Z+1}$$

maps these on straight lines or circles in the w-plane, instead of on focal conics. Hence $a \neq 0$; making $z = \pm i$ in the equation above, it follows that $-A \pm i\sqrt{-C} = 0$ or $A = C = 0$ (since we cannot have

$$e + 1 - a^2 - c \pm 2ai\sqrt{e-c} = 0,$$

which would imply $c = e$ or $z_1^2 = -1$).

Therefore, the left-hand member in (58) has no branch points, while the right-hand member has the branch points $\pm z_1$, since $a \neq 0$, so that this case is impossible.

There finally remains the case

$$(59)\qquad w = z^2,$$

transforming the family of conics (12) with $z_2 = -z_1$ or

$$(60)\qquad \bar{z} = k\frac{\bar{z}_1}{z_1}z + \frac{\bar{z}_1}{z}\sqrt{(k^2-1)(z^2-z_1^2)}$$

and having three parameters k, r, θ, into the conics (with $w_2 = 0$ on account of $E = 0$)

$$(61)\qquad \bar{w} = K\frac{\bar{w}_1}{w_1}w + \frac{1-K}{2}\bar{w}_1 + \frac{\bar{w}_1}{w_1}\sqrt{(K^2-1)w(w-w_1)};$$

by (54a), it follows that $w = w_1$ for $z = z_1$, or by (59) $w_1 = z_1^2$, and writing the conditions that (61) shall result from (60) by means of (59) in the same manner as in the preceding case, we find that these conditions reduce to $K = 2k^2 - 1$. Thus the elements of the conics (61) expressed in terms of those of (60) are

$$(62)\qquad w_1 = z_1^2, \qquad w_2 = 0, \qquad K = 2k^2 - 1.$$

In the particular case $k = 0$, we obtain (44) and (45).

9. **Cases where the conics in the z-plane are ellipses or hyperbolas with both foci fixed.** Placing the foci at $z = \pm 1$, the equation of our conics is (11), and the right-hand member in (34) and (36) becomes

$$\frac{k\sqrt{(k^2-1)(z^2-1)}+(k^2-1)z}{k'z\sqrt{(k^2-1)(z^2-1)}+kk'(z^2-1)}=\frac{\sqrt{k^2-1}}{k'}\cdot\frac{1}{\sqrt{z^2-1}}.$$

Since this is a product of a function of z by a function of t, the case mentioned in the last sentence of § 7 cannot occur here. According to (25) we now write

$$\frac{dZ}{dz}=\frac{1}{\sqrt{z^2-1}},$$

whence

$$(63)\quad Z=\log(z+\sqrt{z^2-1}),\qquad z=\tfrac{1}{2}(e^Z+e^{-Z}),\qquad \sqrt{z^2-1}=\tfrac{1}{2}(e^Z-e^{-Z}),$$

and introducing this in (11), we find

$$e^{\bar{Z}}+e^{-\bar{Z}}=(k+\sqrt{k^2-1})e^Z+(k-\sqrt{k^2-1})e^{-Z},$$

and this quadratic in $e^{\bar{Z}}$ evidently has the two roots

$$e^{\bar{Z}}=(k+\sqrt{k^2-1})e^Z,\qquad (k-\sqrt{k^2-1})e^{-Z}.$$

Hence

$$(64)\qquad \bar{Z}-Z=\log(k+\sqrt{k^2-1}),$$

which represents a real straight line, viz., a parallel to the real axis, when and only when $1>k>-1$, that is, when the conic in the z-plane is a hyperbola, or

$$(65)\qquad \bar{Z}+Z=\log(k-\sqrt{k^2-1})=-\log(k+\sqrt{k^2-1})$$

representing a real straight line, parallel to the imaginary axis, when and only when $k>1$, the conic being then an ellipse. Changing Z into iZ in (63), we find

$$(66)\qquad z=\cos Z,$$

transforming the ellipses and hyperbolas into parallels to the real and imaginary axes in the Z-plane respectively.*

* Writing $Z=\log w$ in (63), so that

$$z=\frac{1}{2}\left(w+\frac{1}{w}\right)$$

it follows that the ellipses and hyperbolas with foci at $z=\pm 1$ are transformed into the circles with center at the origin and straight lines through the origin respectively in the w-plane. This is readily verified from first principles, thus: We have

$$z\pm 1=\frac{1}{2}\left(\sqrt{w}\pm\frac{1}{\sqrt{w}}\right)^2,$$

and for $w=\rho e^{\vartheta i}$,

$$\begin{aligned}|z\pm 1|&=\tfrac{1}{2}\,|\rho^{\frac{1}{2}}e^{\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{-\frac{1}{2}\vartheta i}|^2\\&=\tfrac{1}{2}(\rho^{\frac{1}{2}}e^{\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{-\frac{1}{2}\vartheta i})(\rho^{\frac{1}{2}}e^{-\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{\frac{1}{2}\vartheta i})\\&=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\pm\cos\vartheta,\end{aligned}$$

10. **Cases where the conics in the z-plane are parabolas with fixed focus.** Placing the origin at the focus, the equation of our parabolas is according to (14):

$$\bar{z} = e^{-2\theta i}z - pe^{-\theta i} + 2ie^{-\frac{1}{2}\theta i}\sqrt{pz},$$

which may also be written

(67) $$e^{\frac{1}{2}\theta i}\sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i}\sqrt{z} = i\sqrt{p},$$

and the right-hand member of (34) and (36) is, by (21)

$$\frac{\dfrac{\partial \bar{z}}{\partial z}}{\dfrac{\partial \bar{z}}{\partial t}} = \frac{\dfrac{\partial \sqrt{\bar{z}}}{\partial z}}{\dfrac{\partial \sqrt{\bar{z}}}{\partial t}},$$

or using (67),

(68) $$\frac{1}{-2i\theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}.$$

Consider first the cases where this expression is the product of a function of z by a function of t, namely, $\theta' = 0$ and

(69) $$e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p') = -2i\sqrt{c_1}\theta',$$

where c_1 is a constant and $\theta' \neq 0$. In the first case, $\theta = \text{const.}$, the axis of the parabolas (67) is fixed and may be taken as the real axis, so that $\theta = 0$. Now (25) and (68) give

$$\frac{dZ}{dz} = \frac{1}{2\sqrt{z}},$$

and we have the result that

(70) $$Z = \sqrt{z}$$

transforms the one-parameter family of parabolas with focus at the origin and the real axis as axis

(71) $$\sqrt{\bar{z}} - \sqrt{z} = i\sqrt{p}$$

into the straight lines parallel to the real axis

(72) $$\bar{Z} - Z = i\sqrt{p}.$$

In the second case, the differential equation (69) is readily integrated and gives

$$\sqrt{p} = -i\sqrt{c_1}e^{-\frac{1}{2}\theta i} + c_2 e^{\frac{1}{2}\theta i}$$

whence, for ρ constant, the ellipse

$$|z+1| + |z-1| = \rho + \frac{1}{\rho},$$

and for ϑ constant, the hyperbola

$$|z+1| - |z-1| = 2\cos\vartheta.$$

c_2 being the integration constant, and since $\sqrt{p}$ is real, we must have $c_2 = i\sqrt{\bar{c}_1}$ or

$$\sqrt{p} = -i\sqrt{c_1}e^{-\frac{1}{2}\theta i} + i\sqrt{\bar{c}_1}e^{\frac{1}{2}\theta i}.$$

Now (25), (68) and (69) give

$$\frac{dZ}{dz} = \frac{1}{2}\frac{1}{z + \sqrt{c_1}\sqrt{z}};$$

multiplying z by a constant we may assume that $\sqrt{c_1}$ is real, and we have the result that

$$Z = \log(\sqrt{z} + \sqrt{c_1}) \tag{73}$$

transforms the one parameter family of parabolas with focus at the origin and passing through the fixed point $z = c_1$

$$e^{\frac{1}{2}\theta i}(\sqrt{\bar{z}} + \sqrt{\bar{c}_1}) - e^{-\frac{1}{2}\theta i}(\sqrt{z} + \sqrt{c_1}) = 0 \tag{74}$$

into the straight lines

$$\bar{Z} - Z + \theta i = 0 \tag{75}$$

parallel to the real axis.

When (68) is not the product of a function of z by a function of t, we know from § 4 that w is an algebraic function of z. Consider first equation (34) which becomes by (68)

$$\frac{1}{Lw^2 + Mw + N}\frac{dw}{dz} = \frac{1}{-2i\theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}; \tag{76}$$

giving a constant value to t, making a linear transformation on w according to (32), and remembering that w must be algebraic, we obtain

$$\frac{1}{w}\frac{dw}{dz} = \frac{1}{2}\frac{m}{z + c_1\sqrt{z}},$$

whence

$$w = (\sqrt{z} + c_1)^m$$

with m real and rational. Substituting in (76), we see as in the case of equation (51) that $m = 1$, so that we may make $c_1 = 0$ by moving the origin in the w-plane. From (67) we now immediately have the result that

$$w = \sqrt{z} \tag{77}$$

transforms the two-parameter family (67) of all parabolas with focus at the origin into the two-parameter family

$$e^{\frac{1}{2}\theta i}\bar{w} - e^{-\frac{1}{2}\theta i}w = i\sqrt{p} \tag{78}$$

of all straight lines in the w-plane.

Next we take up equation (36) with at least one variable focus in the w-plane. This focus cannot be a branch point of the left-hand member of (36), since the right-hand member, or (68), has no variable branch points, and we have therefore in the w-plane the case treated in § 7 for the z-plane, so that, according to (42), equation (36) becomes

$$\frac{2Cw}{C'w^2 + E'}\frac{dw}{dz} = \frac{1}{-2i\theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}. \tag{79}$$

Giving t a constant value, and multiplying w and z by suitable constants, we obtain a differential equation which gives upon integration

$$\sqrt{z} + c_2 = (w^2 + c_1)^m$$

and substituting this in (79), we obtain $m = 1$ as before. Replacing $c_1 - c_2$ by $-\sqrt{c_1}$, we have the result that

$$w^2 = \sqrt{z} + \sqrt{c_1}, \tag{80}$$

where we may assume $\sqrt{c_1}$, real, transforms the two-parameter family (67) of all parabolas with focus at the origin into the two-parameter family

$$e^{\frac{1}{2}\theta i}\bar{w}^2 - e^{-\frac{1}{2}\theta i}w^2 = i(\sqrt{p} + 2\sqrt{c_1}\sin\tfrac{1}{2}\theta) \tag{81}$$

of all equilateral hyperbolas with the origin as center in the w-plane.

Finally, consider equation (36) with fixed foci in the w-plane. By § 9, confocal ellipses or hyperbolas in the w-plane would give (68) the form of a product of a function of z by a function of t, which case is excluded by hypothesis. Hence the conics in the w-plane are parabolas, and we may take the common focus as origin. The parameters of these parabolas being P and ϑ, (68) shows that (36) takes the form

$$\begin{aligned}\frac{1}{-2i\vartheta' w + e^{\frac{1}{2}\vartheta i}(\vartheta' P^{\frac{1}{2}} + iP^{-\frac{1}{2}}P')\sqrt{w}}\frac{dw}{dz}& \\ = \frac{1}{-2i\theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}&.\end{aligned} \tag{82}$$

Giving t a constant value, multiplying z and w by suitable constants and integrating the resulting differential equation, we find

$$\sqrt{w} + \sqrt{c_2} = (\sqrt{z} + \sqrt{c_1})^m$$

and substituting in (82), it is seen as before that $m = 1$ (and $c_2 = 0$, c_1 real), whence the result that

$$\sqrt{w} = \sqrt{z} + \sqrt{c_1} \tag{83}$$

transforms the two-parameter family (67) of all parabolas with focus at the origin into itself with a changed parameter p:

(84) $$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i(\sqrt{p} + 2\sqrt{c_1}\sin\tfrac{1}{2}\theta).$$

11. **Summary of results.** Replacing z and w by $az + b$ and $\alpha w + \beta$ respectively, or by $(az + b)/(cz + d)$ and $(\alpha w + \beta)/(\gamma w + \delta)$ when the conics involved are circles, and interchanging z and w when necessary, we may reduce our conformal maps to the following types, arranged according to the number of parameters in the families of conics:

Five parameters (one type):

I. $$w = z;$$

any conic in the z-plane is transformed into the same conic in the w-plane.

Three parameters (two types):

II. $$w = 1/z;$$

any circle or straight line is transformed into a circle or straight line.

III. $$w = z^2;$$

denoting the parameters by k, r and θ, any ellipse ($k > 1$) or hyperbola $(1 > k > -1)$

$$\bar{z} = k\frac{\bar{z}_1}{z_1}z + \frac{\bar{z}_1}{z_1}\sqrt{(k^2 - 1)(z^2 - z_1^2)}$$

with center at the origin and foci at

$$z_1 = re^{\theta i}, \qquad z_2 = -z_1$$

is transformed into the ellipse ($k > 1$) or hyperbola $(1 > k > -1)$

$$\bar{w} = K\frac{\bar{w}_1}{w_1}w + \frac{1 - K}{2}\bar{w}_1 + \frac{\bar{w}_1}{w_1}\sqrt{(K^2 - 1)w(w - w_1)}$$

with $K = 2k^2 - 1$ and foci at

$$w_1 = z_1^2 = r^2e^{2\theta i}, \qquad w_2 = 0.$$

In the special case $k = 0$, the equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin go into the straight lines

$$e^{-2\theta i}w + e^{2\theta i}\bar{w} = r^2.$$

Two parameters, denoted by r, θ or p, θ (four types)

IV. $$w^2 = z^2 - 1;$$

any equilateral hyperbola

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin and foci at $z_1 = re^{\theta i}$, $z_2 = -z_1$ is transformed into the equilateral hyperbola

$$e^{-2\theta i}w^2 + e^{2\theta i}\bar{w}^2 = r^2 - 2\cos 2\theta$$

with center at the origin and foci at

$$w_1 = \sqrt{r^2 - 2\cos 2\theta}\, e^{\theta i}, \qquad w_2 = -w_1.$$

In the special case $r^2 = 2\cos 2\theta$, we have a pair of perpendicular straight lines through the origin in the w-plane.

V. $$w = z^2;$$
any straight line

$$e^{-\frac{1}{2}\theta i}\bar{z} - e^{-\frac{1}{2}\theta i}z = i\sqrt{p}$$

is transformed into the parabola

$$e^{-\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin and at a distance $\frac{1}{2}p$ from the directrix, θ being the angle between the axis of the parabola and the real axis in the w-plane.

VI. $$w = (z^2 - c_1)^2, \qquad \text{where } c_1 = 0 \text{ or } 1;$$
any equilateral hyperbola

$$e^{\frac{1}{2}\theta i}\bar{z}^2 - e^{-\frac{1}{2}\theta i}z^2 = i(\sqrt{p} + 2c_1 \sin \tfrac{1}{2}\theta)$$

with center at the origin and foci at

$$z_1 = e^{\frac{1}{4}(\theta - \pi)i}(\sqrt{p} + 2c_1 \sin \tfrac{1}{2}\theta)^{\frac{1}{2}}, \qquad z_2 = -z_1$$

is transformed into the parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

VII. $$w = (\sqrt{z} - 1)^2;$$
any parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i}\sqrt{z} = i(\sqrt{p} + 2\sin \tfrac{1}{2}\theta)$$

with focus at the origin is transformed into the parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

One parameter, denoted by r, θ or p (nine principal types, the linear composition of each of these with itself or with another giving a total

of forty-five types). There are tabulated below the nine functions which map the one-parameter families of conics specified on the parallels to the real axis in the Z-plane. When $\varphi_1(z)$ and $\varphi_2(z)$ are any two of these functions (identical or different), all maps transforming one-parameter families of conics into others are found by making

$$\varphi_2(w) = A\varphi_1(z) + B,$$

where A and B are constants and A *real*, and then making the linear transformations on z and w indicated at the beginning of this summary.

VIII. $$Z = z;$$

all parallels to the real axis (special case of I).

IX. $$Z = \log z;$$

all straight lines through $z = 0$ (the circles with center at $z = 0$ go into parallels to the imaginary Z-axis).

X. $$Z = i \log z;$$

all circles with center at $z = 0$ (the straight lines through $z = 0$ go into parallels to the imaginary Z-axis).

XI. $$Z = z^2;$$

all equilateral hyperbolas $z^2 + \bar{z}^2 = r^2$ with center at the origin and foci on the real axis (special case of III).

XII. $$Z = \log (z^2 - 1);$$

all equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = 2 \cos 2\theta$$

with center at the origin and foci $\pm \sqrt{2 \cos 2\theta}\, e^{\theta i}$ (which are the end points of the diameter with angle of slope θ of the lemniscate with foci at $+ 1$ and $- 1$). These hyperbolas all pass through the foci of the lemniscate.

XIII. $$Z = \log (z + \sqrt{z^2 - 1}) \qquad \text{or} \qquad z = \tfrac{1}{2}(e^Z + e^{-Z});$$

all hyperbolas with foci at $+ 1$ and $- 1$ (all ellipses with these foci go into parallels to the imaginary Z-axis).

XIV. $$Z = \frac{1}{i} \log (z + \sqrt{z^2 - 1}) \qquad \text{or} \qquad z = \cos Z;$$

all ellipses with foci at $+ 1$ and $- 1$ (all ellipses with these foci go into parallels to the imaginary Z-axis).

XV. $$Z = \sqrt{z};$$

all parabolas $\sqrt{\bar{z}} - \sqrt{z} = i\sqrt{p}$ with focus at the origin and the real axis as axis (special case of V).

XVI. $$Z = \log(\sqrt{z} - 1);$$

all parabolas

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i}\sqrt{z} = 2i \sin \tfrac{1}{2}\theta$$

with focus at the origin and passing through the point $z = 1$. As examples of the possible combinations, we may mention the following: It was shown in § 5 that

$$w = z^k$$

transforms the two families of straight lines through the origin and circles with center at the origin into themselves when k is real and permutes the two families when k is purely imaginary, and $w = z^k$ is obtained by composing VIII with itself:

$$\log w = k \log z$$

when k is real, but VIII with IX

$$\log w = \frac{k}{i} \cdot i \log z$$

when k is purely imaginary.

The map

$$w = \frac{c}{2}(z + \sqrt{z^2 - 1})^m + \frac{1}{2c}(z - \sqrt{z^2 - 1})^m$$

where m is real, transforms the families of all ellipses and all hyperbolas with foci at $+1$ and -1 into themselves, since it is obtained by composing XIII or XIV with itself:

$$\log(w + \sqrt{w^2 - 1}) = m \log(z + \sqrt{z^2 - 1}) + \log c,$$

or

$$\frac{1}{i}\log(w + \sqrt{w^2 - 1}) = m \cdot \frac{1}{i}\log(z + \sqrt{z^2 - 1}) + \frac{1}{i}\log c.$$

ON THE LOCATION OF THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.*

By J. L. Walsh.

1. Introduction: Jensen's Theorem. This paper contains some geometric results concerning the relative positions of the roots of a polynomial and those of its derivative. Although not entirely restricted to real polynomials, and although the cubic is especially treated in detail, most of the results here presented are naturally connected with the following theorem of Jensen's:

If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles.†

For brevity we shall call the circles with which this theorem is concerned *Jensen circles.*

The succeeding developments follow largely from Gauss's theorem that the roots of the derivative are the positions of equilibrium in the field of force due to particles one situated at each root of the original polynomial, each particle repelling as the inverse distance. The derived polynomial has roots not only at the positions of equilibrium but also at the multiple roots of the original polynomial. When we are concerned with real polynomials especially it seems natural to study the field of force due to two particles.

2. The Field of Force due to Two Particles. In the field of force due to particles of the kind described, the force at a point P due to a particle at Q is in direction, magnitude, and sense $Q'P$, where Q' is the inverse of Q in the unit circle whose center is P. The force at P due to k particles is equivalent to k coincident vectors with one terminal at P and the other at the center of gravity of the inverses of the positions of those k particles. In the sequel we shall have frequent occasion to use this fact.

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* Presented to the American Mathematical Society, December 31, 1919.

† This theorem was stated without proof by Jensen, Acta Mathematica, vol. XXXVI (1912), p. 190. Attention was called to it by Professor D. R. Curtiss in an abstract published in the Bulletin of the American Mathematical Society, vol. XXVI, p. 62. No proof of Jensen's theorem has previously been published.

Two recent papers by J. S. Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 27 (1918), pp. 37, 44, contain some results concerning the roots of the derivative of a polynomial, more particularly if all the roots of the original polynomial are real.

According to the previous notation, the force at P due to unit particles at Q and R is in direction, magnitude, and sense $S'P$, where S' is the mid-point of the segment $Q'R'$. Since S' is the harmonic conjugate of the point at infinity with respect to Q' and R', and since cross-ratios are invariant under inversion, it follows that *the force at P due to unit particles at Q and R respectively is equivalent to the force at P due to two coincident particles situated at S, the harmonic conjugate of P with respect to Q and R.* The point S may of course be constructed by ruler and compass; we shall describe the case where Q and R are the points $+ i$ and $- i$. At P construct the tangent to the circle through P, $+ i$, and $- i$. Using as center the intersection of this tangent with the axis of imaginaries describe a circle through P. This circle intersects at the point S the circle through P, $+ i$, and $- i$. An alternate construction is found by noticing that the lines joining the origin with P and S are symmetric respecting the coördinate axes. If P is on either coördinate axis a construction is used which differs slightly from either of these but which is easily devised.

We obtain immediately some results concerning the field of force. It is symmetric respecting each coördinate axis; at a point on either axis the force is directed along that axis. At any point on the unit circle whose center is the origin, the force is horizontal. Inside that circle but above the axis of reals, the force has a component vertically downward. Outside that circle but above the axis of reals the force has a component vertically upward. The line of action of a force always cuts the axis of imaginaries between the points $+ i$ and $- i$.

On any circular arc bounded by the points $+ i$ and $- i$, the force has a minimum on the axis of reals. On the unit circle whose center is the origin, the minima occur at $+ 1$ and $- 1$, where the force is of magnitude 1.

The relation between P and S is reciprocal; when expressed in terms of complex variables as coördinates that relation is linear. Hence when one of the points P, S moves in a circle so does the other.

We shall now proceed to determine the lines of force. The field of force is given by

$$\frac{f'(\bar{z})}{f(\bar{z})} = \frac{2(x - iy)}{(x - iy)^2 + 1},$$

and this leads to the differential equation

$$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

The solution of this gives us the lines of force

$$x^2 - y^2 + 1 = Cxy,$$

(together with $xy = 0$) which are equilateral hyperbolas having the origin as center and passing through the points $+ i$ and $- i$.

What is the locus of points such that the lines of action of the forces there all pass through a fixed point $(a, 0)$*? The line of action of the force at (x_1, y_1) is given by

$$x_1x - y_1y + 1 = \frac{C}{2}(x_1y + y_1x).$$

We have also the equations

$$ax_1 + 1 = \frac{C}{2}ay_1, \qquad x_1^2 - y_1^2 + 1 = Cx_1y_1,$$

from which we obtain

$$\left(x_1 + \frac{1}{a}\right)^2 + y_1^2 = 1 + \frac{1}{a^2} \qquad (a \neq 0),$$

which is a circle easily constructed by ruler and compass and which passes through the points $+ i$ and $- i$.

The results of this paragraph have been deduced on the assumption that we have merely two single particles, whose distance apart is 2. It is obvious what are the results for k particles at each of two points with any distance between them.

3. Some Immediate Results. A proof of Jensen's theorem is now evident. At a point not on the axis of reals nor on or within any Jensen circle, the force has a vertical component in direction away from the axis of reals—this is true of the force due to any pair of conjugate imaginary roots of $f(z)$, also true of the force due to any real root of $f(z)$, so it is true of the resultant. Hence such a point cannot be a position of equilibrium. Such a point cannot be a multiple root of $f(z)$ and hence cannot be a root of $f'(z)$.† We may add that a point of a Jensen circle not on the axis of reals cannot be a root of $f'(z)$ unless it is on or within another Jensen circle or is a multiple root of $f(z)$.

Jensen's theorem can be generalized as follows:

If all the roots of a polynomial $f(z)$ not on a line L nor one of a pair situated symmetrically with respect to L lie on one side of L, then on the opposite side of L there are no roots of $f'(z)$ except on or within circles whose diameters are the segments joining those symmetric roots.

This theorem and likewise Jensen's theorem are limiting cases of the following:

* For a point not on the axis of reals we are in general led to a cubic equation.

† A more immediate but less elegant proof can be given by the method of inversion indicated at the beginning of § 2.

If $f(z)$ is a polynomial such that every root not in nor on a circle C is one of a pair of roots symmetric with respect to C, then all the roots of $f'(z)$ lie in or on C and in or on circles whose diameters are the segments joining those pairs of symmetric roots.

The proof here is similar to the proof of Jensen's theorem—the force at a point P external to C has an outward component along the line from the center of C to P. Similarly, it is readily shown by the geometric construction given that the force at P (external to the other circles considered) due to any pair of roots symmetric respecting C has also an outward component along that line.

There are an unlimited number of theorems that can be written down immediately. We give a few more examples.

If a circle C contains all but P and Q, two equal roots of a polynomial $f(z)$, contains neither of them, but has its center on the line PQ and not on the segment PQ, then no roots of $f'(z)$ lie in that semicircular region bounded by the perpendicular bisector of PQ and that half of the circle on PQ as a diameter which is nearer C. There are also no roots of $f'(z)$ outside of the circle on PQ as a diameter and in that half plane bounded by the perpendicular bisector of PQ which does not contain C.*

The limiting case of this theorem is also true—C is replaced by a straight line.

If $f(z)$ is a real polynomial having equal roots at $+ i$ and $- i$, if there is no other root of $f(z)$ whose abscissa is less than $\alpha > 1$, and if the point $z = 1$ is interior to no Jensen circle, then there is no non-real root of $f'(z)$ in the circle whose center is $- 1/\alpha$ and which passes through the points $+ i$ and $- i$.

In this proof we need consider only that part of the last circle lying to the right of the axis of imaginaries. The line of action of the force at any point inside that circle due to the particles at $+ i$ and $- i$ cuts the axis of reals to the left of the point $x = \alpha$. The line of action of the force due to the remaining particles (whether these be real or in conjugate imaginary pairs) cuts that axis in a point not to the left of the point $x = \alpha$.

All of the theorems stated in this paragraph may give a closer idea of the location of the roots of the derived polynomial than does Lucas's theorem that the roots of $f'(z)$ lie in any convex polygon enclosing the roots of $f(z)$. Since the location of the roots of $f'(z)$ does not in general depend on the solution of a quadratic equation, it is not to be expected that the ruler-and-compass constructions used here would determine

* Varying the roots of $f(z)$ which are on or in C and finally allowing them to coalesce at a point in C on PQ (this method is used frequently later) shows that the other semicircular region with these boundaries has in its interior precisely one root of $f'(z)$.

precisely the location of the roots. In exceptional cases, however, that location can be found.*

If a polynomial $f(z)$ has only four roots, and if these are equal in multiplicity and located at the vertices of a rectangle $ABCD$, then the circles whose diameters are the two longer sides of the rectangle, AB and CD, intersect in roots of $f'(z)$. For the force at each of those points due to each of the pairs of particles (A, B) and (C, D) is in direction parallel to the shorter sides of the rectangle, and the two forces at each point are equal in magnitude and opposite in direction. Symmetry shows that a third root of $f'(z)$ lies at the center of the rectangle. If the rectangle becomes a square, the center is a three-fold root of $f'(z)$.

Another example of finding the explicit location of the roots of $f'(z)$ is given in the last paragraph of this paper.

4. A Theorem Complementary to Jensen's Theorem. The question of how many roots of $f'(z)$ are situated in a Jensen circle, and how many in the intervals of the axis of reals readily suggests itself. An answer is given in Grace's theorem, which indeed is true for both real and non-real polynomials:

If a polynomial $f(z)$ of degree n has roots at $+i$ and $-i$, there is at least one root of $f'(z)$ on or in the circle whose center is the origin and radius $\cot \pi/n$.†

This maximum value for the modulus of the root of $f'(z)$ which is nearest the origin is actually assumed when the roots of $f(z)$ are the vertices of a regular polygon of n sides whose center is $\cot \pi/n$.

A theorem other than this can also be proved for real polynomials, and which may give more definite results than Grace's theorem; first we prove:

In the interior of any interval of the axis of reals containing no root of $f(z)$ and exterior to Jensen's circles, there is at most one root of $f'(z)$.

The theorem refers merely to the *interior* of an interval, so it is sufficient to prove the theorem assuming that neither extremity is a root of either $f(z)$ or $f'(z)$. Consider the interval $\alpha \leqq x \leqq \beta$, and suppose first that the forces at α and β are in the same direction—toward the right for definiteness. Move all the roots of $f(z)$ whose abscissas are greater than β horizontally and continuously to the right, and allow them to become infinite. The roots of $f'(z)$ also move continuously (at least when we consider the stereographic projection of the plane) and one or more may become infinite. The forces at α and β continually increase in magnitude

* There is of course the trivial case where M and N are respectively m- and n-fold roots of $f(z)$. There being no other roots, the point dividing the segment MN in the ratio $m : n$ is a root of $f'(z)$.

† Proceedings of the Cambridge Philosophical Society, vol. XI (1901–02), p. 352. Proved later but independently by Heawood, Quarterly Journal of Mathematics, vol. XXXVIII (1907), p. 84.

and are never zero. Hence throughout the process there are a fixed number of roots in the interval—a number which at the end of the process is evidently zero.

Secondly we consider the case that the force at α is directed toward the right, and the force at β toward the left. Here there is evidently at least one root in the interval. Let μ_1 and μ_2 be any pair of conjugate imaginary roots of $f(z)$ whose common abscissa is greater than β. Allow them to move continuously toward the right, always remaining conjugate imaginary, along the circle through μ_1, μ_2, and α, and finally to coincide on the axis of reals. This motion keeps constant the force at α and continually increases the magnitude of the force at β. Treat in this manner all the pairs of conjugate imaginary roots of $f(z)$ whose abscissas are greater than β, and correspondingly treat all the conjugate imaginary roots of $f(z)$ whose abscissas are less than α, moving them so that the force at β is kept constant. During the whole motion the roots of $f'(z)$ vary continuously, none can enter or leave the interval, and hence at the beginning there was precisely one root of $f'(z)$ in the interval.

Thirdly, it is conceivable that the force at α should be directed toward the left and that at β toward the right. This means that the force at α due to the particles whose abscissas are greater than β is greater than the force at β due to those particles, which is impossible.

From the theorem just proved and by similar methods we shall deduce:

If a Jensen circle has on or within it k roots of $f(z)$ and is not interior to nor has a point in common with any exterior Jensen circle, then it has on or within it not more than $k + 1$ nor less than $k - 1$ roots of $f'(z)$.

Denote by μ and ν $(\mu < \nu)$ the intercepts of this circle C with the axis of reals. If the forces at μ and ν are respectively directed toward the right and left, and if the particles whose abscissas are less than μ are moved by translation to the left and to infinity, one and only one root of $f'(z)$ will issue from C, and that toward the left. If the particles whose abscissas are greater than ν are translated horizontally to the right and to infinity, one and only one root of $f'(z)$ will issue from C, and that toward the right. Finally $k - 1$ roots of $f'(z)$ remain in or on C, and therefore the original number was $k + 1$. If the forces at μ and ν are respectively toward the left and right, and if the particles exterior to C are translated horizontally to infinity, although during the motion one root of $f'(z)$ may enter C, it will eventually issue from C. The final number of roots on or within C is the same as the original number, $k - 1$.* In a similar manner it is

* Immediately we obtain the theorem: *If $f(z)$ is a real polynomial with two simple roots at $+i$ and $-i$, with m roots whose abscissas are greater than $m + 1$, n roots whose abscissas are less than $-n - 1$, and no other roots, then $f'(z)$ has one real root and no other root in the unit circle whose center is the origin.* Cf. the theorem of § 6.

shown that if the forces at μ and ν are in the same sense, there are on or within C precisely k roots of $f'(z)$. The cases where μ or ν or both are roots of $f'(z)$ are easily treated.

5. A Theorem Related to Jensen's Theorem. Jensen's theorem gives a configuration—the Jensen circles and the axis of reals—in which all roots of $f'(z)$ must lie, and it is easy to see that no more restricted locality will satisfy the conditions of the theorem. First, any point of the axis of reals may be a multiple root of $f(z)$ and hence a root of $f'(z)$. Second, we may have a root of $f'(z)$ as near as desired to a point ρ interior to a Jensen circle in any preassigned configuration. Let the line of action of the force at ρ due to the particles μ and ν determining the Jensen circle intersect the axis of reals at a point σ. If the distance from ρ to ρ', its harmonic conjugate respecting μ and ν, is commensurable with the distance from ρ to σ, then neglecting the other particles in the field, by a proper choice of the multiplicities of the roots of $f(z)$ at μ, ν, and σ, we can make ρ a root of $f'(z)$. If the two distances are not commensurable, and taking account of the other particles in the plane, we can make the multiplicities of the roots of $f(z)$ at μ, ν, and σ very large in comparison with the other roots of $f(z)$, and in such ratio that there is a root of $f'(z)$ as near to ρ as desired.

This reasoning refers to a preassigned configuration rather than a preassigned polynomial; it is of course impossible if the degree of $f(z)$ is limited; by considering polynomials of a fixed degree we may expect to obtain some results concerning a region more restricted.

Considering the polynomial

$$f(z) = (z^2 + 1)(z - \alpha)^{n-2},$$

where α is real, we shall prove that the non-real roots of $f'(z)$ lie on the circle whose center is the origin and radius $\sqrt{(n-2)/n}$. We eliminate α from the equations obtained from the real and pure imaginary parts of the equation

$$(z - \alpha)^{-n+3} f'(z) = n(x + iy)^2 - 2\alpha(x + iy) + n - 2 = 0, \qquad y \neq 0;$$

$$x^2 + y^2 = \frac{n-2}{n}.$$

When α is large and positive, one of the roots of $f'(z)$ different from α is near the origin and the other near the point $2\alpha/n$. As α decreases, the former root moves to the right, while the latter moves to the left. The two roots coalesce at $\sqrt{(n-2)/n}$ when $\alpha = \sqrt{n(n-2)}$. As α continues to decrease, the roots move on the circle already determined, remaining conjugate imaginary, and when $\alpha = 0$ those two roots are $\pm i\sqrt{n(-2)}/n$. The path of the roots when α further decreases is found from symmetry.

We shall extend this result and prove:

If $f(z)$ is a real polynomial of degree n whose roots are all real except simple roots at $+i$ and $-i$, then all the non-real roots of $f'(z)$ lie in or on the circle whose center is the origin and radius $\sqrt{(n-2)/n}$.

In the proof, we first notice—this is in the nature of a lemma—that if the force at a point P is in direction along a line l and due to k particles on a line λ, then the force is not greater than the force at P due to k coincident particles situated at the intersection of l and λ. The lemma is proved by the method of inversion previously described (§ 2).

We next consider the force at points along the arc of a circle bounded by the points $+i$ and $-i$, the arc intersecting the axis of reals at β, between the points $\sqrt{(n-2)/n}$ and 1. The lines of action of the force at points of the arc due to the particles at $+i$ and $-i$ all pass through the point $2\beta/(1-\beta^2)$. The force at β due to $n-2$ particles at $2\beta/(1-\beta^2)$ is less than the force at β due to the two particles at $+i$ and $-i$, for if we have

$$\frac{n-2}{\dfrac{2\beta}{1-\beta^2}-\beta} \geqq \frac{2}{2\dfrac{1+\beta^2}{2\beta}},$$

are led to

$$\frac{n-2}{n} \geqq \beta^2,$$

which is contrary to our assumption. The force at any point on the arc considered due to the particles at $+i$ and $-i$ increases in magnitude as we move from the axis of reals toward $+i$ or $-i$. Moreover the force due to $n-2$ particles at $2\beta/(1-\beta^2)$ decreases, so no point on the arc can be a position of equilibrium when the $n-2$ particles coalesce. From the lemma, then, no such point can be a position of equilibrium in any other case.

The treatment of the arcs of circles which have a point in common with the circle whose center is the origin and radius $\sqrt{(n-2)/n}$ can readily be made, noting as before that the force due to the particles at $+i$ and $-i$ increases as we move away from the axis of reals, while the force due to the $n-2$ coincident particles decreases. This completes the proof.

6. Sufficient Conditions for the Reality of the Roots of $f'(z)$. When all the roots of $f(z)$ except two non-real roots are sufficiently removed from the latter, the roots of $f'(z)$ in the corresponding Jensen circle are real. This paragraph gives a theorem containing sufficient conditions for the reality of the roots of $f'(z)$, which theorem is stated simply to concern one Jensen circle, but may of course be applied to several in order.

If $f(z)$ is a real polynomial with simple roots at $+i$ and $-i$, m roots

whose abscissas are greater than $\sqrt{m(m+2)}$, *n roots whose abscissas are less than* $-\sqrt{n(n+2)}$, *and with no other roots, then* $f(z)$ *has precisely one root in the interval* $(-\sqrt{n(n+2)}, \sqrt{m(m+2)})$ *and no non-real root in the Jensen circle whose center is the origin.*

The degenerate cases here are first $m = 0$, $n \neq 0$, and all the n roots concentrated at $-\sqrt{n(n+2)}$, in which case $f'(z)$ has a double root at $-\sqrt{n(n+2)}$; and second $m \neq 0$, $n = 0$, all the m roots concentrated at $\sqrt{m(m+2)}$, in which case $f'(z)$ has a double root at $\sqrt{m(m+2)}$. In either of these cases we make the convention that simply one of those roots belongs to the interval mentioned.

In any non-degenerate case, the force at $\sqrt{m(m+2)}$ is directed toward the right, for otherwise we have the force at that point due to the particles at $+i$ and $-i$ less than the force at that point due to the m particles:

$$\frac{2}{\sqrt{\dfrac{m+2}{m}}+\sqrt{\dfrac{m}{m+2}}} < \frac{m}{\sqrt{m(m+2)}-\sqrt{\dfrac{m}{m+2}}}, \qquad m < m,$$

which is absurd. Similarly, the force at $-\sqrt{n(n+2)}$ is directed toward the left, so the interval of the theorem contains at least one root of $f'(z)$.

Suppose neither of the points $+1$ and -1 to lie on or within any Jensen circle except of course the unit circle C whose center is the origin. Then C contains at least one root and not more than three roots of $f'(z)$. In fact, by considering the forces at the points $+1$ and -1 we immediately determine from the results of § 4 whether C contains one, two, or three roots, and it is then evident that all those roots are real.

We shall now prove that under no circumstances consistent with our hypothesis can any point of C except $+1$ and -1 be a root of $f'(z)$. First suppose a point of C in the first quadrant to lie in or on one of the Jensen circles pertaining to the m roots. At such a point (x, y), the horizontal component of the force due to the two particles at $+1$ and -1 is greater than the horizontal component of the force due to the m particles. Assuming the contrary, we must have

$$\frac{2}{2x} \leqq \frac{2}{2(\sqrt{m(m+2)}-1)} + \frac{m-2}{\sqrt{m(m+2)}-1},$$

$$x \geqq \frac{\sqrt{m(m+2)}-1}{m-1} > 1,$$

which is impossible. The proof just given is also valid for the point $+1$,—if that point is on or within one of the Jensen circles belonging to the m roots, it is not a root of $f'(z)$.

It is conceivable, secondly, that a point $P:(x, y)$ in the first quadrant and on C should lie exterior to all the Jensen circles pertaining to the m roots, and yet should lie interior to one or more of the Jensen circles pertaining to the n roots, and should be a root of $f'(z)$. We shall prove the impossibility of this, roughly, as follows: such a root of $f'(z)$ must be near the point $+1$, for as we move upward from that point along C the horizontal force at P due to the particles at $+i$ and $-i$ becomes greater and eventually exceeds the force at P due to the m particles. The force due to the m particles is such that the vector representing the total force at P due to the m particles and the particles at $+i$ and $-i$ is inclined to the horizontal at a comparatively steep angle. In order for the force at P due to two or more of the n particles to be inclined at that same angle, P must be quite near the center of the corresponding Jensen circle, which proves to be impossible.

Consider the slope of the line of action of the force due to the two particles at $+i$ and $-i$ and to the m particles (the force always with a component toward the left),—this slope is numerically least when the m roots are all concentrated at $\sqrt{m(m+2)}$. For invert the configuration in the unit circle whose center is P, except that the point $(-x, y)$ is to be inverted into a point Q by means of a circle whose center is P and radius $\sqrt{-1}$. When we replace the m particles of the theorem by m coincident particles, the terminal of the vector corresponding is seen to lie in or on the boundary of the sector of a circle, which sector is bounded by the line through P and $\sqrt{m(m+2)}$, and by the circle which is the inverse (regarding the unit circle whose center is P) of the reflection of that line in the axis of reals. The point of contact of a tangent from Q to that circle cannot lie between the point which is the inverse of $\sqrt{m(m+2)}$ and the intersection of the line PQ with the circle. This fact is proved most easily, perhaps, by inverting the inverse figure (including Q) again in the unit circle whose center is P. The details are omitted here, but this completes the proof that the slope is numerically least when the m roots are concentrated at $\sqrt{m(m+2)}$.

If all the m roots are located at $\sqrt{m(m+2)}$, the total force at P due to the m particles and to the particles at $+i$ and $-i$ has a slope numerically equal to

$$\frac{mxy}{x\sqrt{m(m+2)^3} - mx^2 - m(m+2) - 1},$$

assuming that the force has a component toward the left. If this quantity is less than

$$\frac{my}{\sqrt{m(m+2)^3} - m - m(m+2) - 1},$$

we shall have

$$- mx - m(m+2)x - x < - mx^2 - m(m+2) - 1,$$

$$mx(1-x) + m(m+2)x + x > m(m+2) + 1,$$

which inequality is false for $x < 1$.

We turn now to consideration of the force due to one pair of the n particles, using running coördinates (ξ, η). The slope of the line of action of the force at (ξ, η) is

$$\frac{\eta}{\xi} \cdot \frac{\xi^2 + \eta^2 - 1}{\xi^2 + \eta^2 + 1},$$

and the locus of points at which the force has the slope $-\mu$ is given by

$$(\eta + \mu\xi)(\xi^2 + \eta^2) - (\eta - \mu\xi) = 0.$$

All points in the first quadrant and interior to the corresponding Jensen circle at which the slope of the force is numerically greater than μ lie above the line $\eta - \mu\xi = 0$. For the point P considered above, we have

$$\mu \leqq \frac{y}{\sqrt{n(n+2)}} < y.$$

Then if P is a position of equilibrium we must have

$$y > \frac{my}{\sqrt{m(m+2)^3} - m - m(m+2) - 1}, \qquad \sqrt{m(m+2)} > m + 1,$$

which is impossible.

We have therefore proved that no point of the circle C except $+1$ or -1 can be a root of $f'(z)$, and that $+1$ is not a root if it lies on or within one of the Jensen circles pertaining to the m roots, nor -1 if it lies on or within one of the Jensen circles pertaining to the n roots.

If the Jensen circle of any pair of the m roots incloses or intersects the unit circle whose center is the origin, continuously move those roots toward the right along the circle joining them with the point -1, and move them until the Jensen circle cuts the axis of reals slightly to the right of the point $+1$, and so that there is no root of $f'(z)$ on the axis of reals between the point of intersection and the point $+1$. During this motion the force at the point -1 is constant, so there is no change in the number of roots inside C. Similarly move any pair of the n roots whose Jensen circle incloses or cuts the unit circle, keeping the force constant at the point $+1$. In this final position, the forces at the points $+1$ and -1 are in the same direction as were the forces in the initial position, never having changed sense. In the final position—which is of course the

initial position so far as concerns Jensen circles not inclosing nor having a point in common with C—the circle C contains one, two, or three roots of $f'(z)$ according as the forces at $+1$ and -1 are both, one, or neither directed away from the origin. The forces at the points $\sqrt{m/(m+2)}$, $-\sqrt{n/(n+2)}$ are initially and finally directed away from the origin, so it is clear where the one, two, or three roots of $f'(z)$ lie inside the circle C,—and in the same intervals in the initial and final positions. The reader will readily take up the possibility that one or both of the points $+1$ and -1 may be a root of $f'(z)$, and this will complete the proof of the theorem.

It is to be noticed that the intervals given in the hypothesis of the theorem are the smallest which will insure the reality of the roots of $f'(z)$. For if we allow, for example, an abscissa smaller than $\sqrt{m(m+2)}$ for the m roots, we can concentrate them at the point nearest the origin and remove the n roots so far (by changing either their abscissas or ordinates) that their influence in the field of force is as small as desired. Hence $f'(z)$ will have two non-real roots in the circle C.

7. The Reality and Non-Reality of the Roots of $f'(z)$. A General Theorem. Having derived a sufficient condition for the reality of roots of $f'(z)$, we shall now derive a sufficient condition for the non-reality of roots. A number of results will then be collected into a general theorem.

If $f(z)$ is a real polynomial of degree $n > 2$ with simple roots at $+i$ and $-i$, and if all the other roots of $f(z)$ are interior to the interval $(0, \sqrt{n(n-2)})$, then $f'(z)$ has precisely two non-real roots.

First, $f'(z)$ can have no root interior to a finite interval of the axis of reals bounded by the origin and a root of $f(z)$ but containing no root of $f(z)$. For consideration of the forces at such a point x due respectively to the particles at $+i$ and $-i$ and the particles on the axis of reals would lead to the inequalities

$$\frac{2}{x+\frac{1}{x}} > \frac{n-2}{\sqrt{n(n-2)}-x},$$

$$0 > \left(\sqrt{nx} - \sqrt{\frac{n-2}{x}}\right)^2.$$

Second, there can be no more than one root of $f'(z)$ interior to an interval of the axis of reals bounded by two roots of $f(z)$. If there were, by moving to the left the root of $f(z)$ which is the right-hand boundary of that interval, eventually at least two roots of $f'(z)$ must become imaginary. For when a k-fold root and an l-fold root of $f(z)$ coalesce, the point is a $(k+l-1)$-fold root of $f'(z)$.

It will now be shown that no point (x, y) interior to the circle whose center is the origin and radius unity and whose ordinate is positive and less than $\frac{1}{2}$ can be a root of $f'(z)$, assuming that there is at least one root of $f(z)$ in the interval of the theorem and whose abscissa is less than x. The vertical component of the force at (x, y) due to the particle in that position is not less than the component for a particle at the origin:

$$\frac{y}{x^2 + y^2}.$$

The vertical component of the force at (x, y) due to the two particles at $+ i$ and $- i$ is numerically

$$\frac{2y(1 - x^2 - y^2)}{(x^2 + y^2)^2 + 2(x^2 - y^2) + 1}.$$

Assuming that (x, y) is a root of $f'(z)$, we have

$$\frac{2y(1 - x^2 - y^2)}{(x^2 + y^2)^2 + 2(x^2 - y^2) + 1} \geqq \frac{y}{x^2 + y^2}, \qquad 4y^2 - 1 \geqq 3(x^2 + y^2)^2,$$

which is impossible if $y < \frac{1}{2}$.

This completes the proof that there is not more than one root of $f'(z)$ interior to any interval of the axis of reals bounded by two roots of $f(z)$, and hence there are precisely two non-real roots of $f'(z)$. It may be added that the entire argument remains valid if in the plane there are k particles of positive abscissas none of whose Jensen circles includes nor has a point in common with the unit circle whose center is the origin—this last-named circle contains precisely two non-real roots of $f'(z)$.

We shall summarize a number of the previous results in the theorem:

If $f(z)$ is a polynomial of degree n with simple roots at $+ i$ and $- i$, and if all the remaining $n - 2$ roots are real and—

(1) *concentrated at $\sqrt{n(n-2)}$, $f'(z)$ has a double root at $\sqrt{(n-2)/n}$*

(2) *with abscissas not less than $\sqrt{n(n-2)}$, $f'(z)$ has all its roots real, precisely one of which lies in the interval $(0, \sqrt{(n-2)/n})$.*

(3) *with abscissas non-negative but less than $\sqrt{n(n-2)}$, $f'(z)$ has precisely two non-real roots.*

(4) *with abscissas unrestricted, the non-real roots of $f'(z)$ lie on or within the circle whose center is the origin and radius $\sqrt{(n-2)/n}$.*

(5) *with abscissas unrestricted but coincident, the non-real roots of $f'(z)$ lie on that circle.*

8. Variation of the Roots of a Real Cubic. This paragraph considers how the roots of the derivative of a real cubic vary when one of the roots of the cubic varies, and also how the roots of a real cubic may vary so that

the roots of the derivative are fixed. We shall have frequent occasion to use the well-known theorem that the roots of a polynomial and those of its derivative have a common center of gravity.

There has already been described (§ 5, $n = 3$) the variation of the roots of the derivative of a real cubic with two fixed non-real roots when the third (real) root of the cubic varies. Of course the variation of the roots of the derivative when the real root is fixed and the two non-real roots move horizontally is essentially identical with that. When the real root is fixed and the two non-real roots move in another manner, the motion of the roots of the derivative is easily determined. For example, if the two non-real roots of the cubic move in a vertical line, and if the roots of the derivative are not real they also move in a vertical line.

For the sake of completeness we consider also a cubic whose roots 1, -1, α are all real. When α is very large and positive, the two roots of the derivative are approximately at the origin and the point $2\alpha/3$. When α decreases, both roots move to the left, and when $\alpha = 1$ these roots have reached the points $-\frac{1}{3}$ and 1 respectively. When α continues to decrease, the roots continue their motion to the left, and when $\alpha = 0$ these roots are at $\pm \frac{1}{3}\sqrt{3}$. For negative values of α the location of the roots is obtained from symmetry.

When the cubic has two coincident roots at 0 and a third root at α, the roots of the derivative are at 0 and $2\alpha/3$.*

We shall now consider what real cubics have given fixed points as the roots of their derivatives, first choosing those points at $+i$ and $-i$. The cubic itself must be of the form

$$f(z) = z^3 + 3z + C,$$

and therefore we have to study the (C, z) transformation. We shall describe the result rather in terms of the variation of α, the one real root of $f(z)$. When $\alpha = 0$, the other two roots of $f(z)$ are at the points $\pm \sqrt{3}i$. When α moves to the right or left, these other roots move toward the left or right, one on each branch of the hyperbola

$$x^2 - \frac{y^2}{3} = -1.$$

The common abscissa of the two non-real roots of $f(z)$ is always $-\alpha/2$. This completely determines the motion.

Suppose two fixed points $+1$ and -1 are the roots of the derivative

* If $f(z)$ is a non-real cubic two of whose roots are fixed while the third traces a line bisecting their segment, the roots of $f'(z)$ trace a cubic curve having that line as asymptote. Only the degenerate cases of the cubic curve have been considered in detail here.

of a real cubic. How do the roots of the cubic vary? If those roots are all real, there is one of them in each of the intervals $(-\infty, -1)$, $(-1, +1)$, $(+1, +\infty)$, with the obvious convention regarding roots at the ends of these intervals. When one real root of the cubic is $\alpha = 2$, the other roots coalesce at -1. When α moves to the left, the other roots move along the axis of reals, one to the right and one to the left. The former coincides with α at the point 1, when the latter has reached the point -2. As α further moves to the left, the former root moves from 1 toward the right, while the latter moves from -2 also to the right. When $\alpha = 0$, these roots are at $\pm\sqrt{3}$. When α reaches -1, the root moving from the left coincides with it, whereas the other root is at the point 2. As α moves to the left from the point -1, the other two roots move toward each other, and coalesce at $+1$ when $\alpha = -2$. As α continues its motion to the left, those roots move along the right-hand branch of the hyperbola

$$x^2 - \frac{y^2}{3} = 1.$$

The common abscissa of those two imaginary roots is always $-2\alpha/3$. Symmetry now gives us a complete discussion of the situation.

When the real cubic has two real coincident roots at the origin, the roots of the cubic lie one on each of the lines

$$y = 0, \qquad y = \sqrt{3}x, \qquad y = -\sqrt{3}x.$$

The three roots lie always at the vertices of an equilateral triangle whose center is the origin.

9. Ruler-and-Compass Construction for the Roots of the Derivative of a Cubic. It is to be expected that the roots of the derivative of a cubic have some interesting properties relative to the triangle whose vertices are the roots of the original cubic. In fact, it has been proved that these two points are the foci of the maximum ellipse which can be inscribed in that triangle, which ellipse touches the sides of the triangle at their mid-points.* We shall use this property to give a ruler-and-compass construction for the roots of the derivative. Of course those roots depend on the solution of a quadratic, so it is known a priori that they can be located by ruler and compass.

Let A, B, and C be the roots of the original cubic, let F be the mid-point of AB, and let the intersection of the medians of the triangle be M.

* This seems first to have been proved by F. J. van den Berg, Nieuw Archief von Wiskunde, 1882, 1884, 1888. That reference is not available to the present writer, but is given indirectly by E. Cesàro, Periodico di Mat., vol. XVI (1900–01), p. 81. See also M. Bôcher, these Annals, vol. VII (1892), p. 70; Grace, l. c.; Heawood, l. c.

Let a line through M parallel to AC intersect AB in D. Then MD and MB are in direction conjugate diameters of the ellipse. Determine the length MG such that

$$\overline{MG}^2 = DF \cdot FB,$$

which construction is readily made. Lay off this length from M on a line through M parallel to AB. Then MF and MG are in direction and length conjugate diameters of the ellipse. For if any tangent meets two conjugate semidiameters of an ellipse, the rectangle under its segments is equal to the square of the parallel semidiameter.*

Knowing in direction and magnitude two conjugate semidiameters of the ellipse, we can find the foci.† From F draw FN perpendicular to MG and produce FN its own length to H. Join MH, and on MH as diameter describe a circle whose center is denoted by K. Join FK, cutting the circle in P and Q. Lay off on MP, $MX = FQ$ and on MP lay off $MY = FP$. Then MX and MY are the axes of the ellipse; the foci may be found as the intersection with MX of a circle whose center is Y and radius MX.*

This construction can be greatly simplified in some special cases, notably if the polynomial is real or more generally if the triangle ABC is isosceles. If ABC is an equilateral triangle, the intersection of the medians is a double root of $f'(z)$. If $AB < BC = CA$, the circle with M as center and MF as radius is the major auxiliary circle of the ellipse. Let this circle cut AC in the points S and T. If R is the mid-point of AC, lines through R parallel to MS and MT respectively cut CM in the foci. For the length of a line through the center parallel to either focal radius vector and terminated by the tangent is the semi-major axis.§ If $AB > BC = CA$, denote by V the intersection of RM with AB. Then FV is the semi-major axis, so we can complete the construction as before.

There seems to be no obvious construction applicable when the points A, B, C are collinear, but we can get a rather simple procedure with the aid of the equations involved. The center of gravity of the three roots is easily found by ruler and compass, so we can choose that point as the

* See, e.g., Casey, Analytical Geometry (1893), p. 231. In fact we may simply lay off $MG : AB = 1 : \sqrt{12}$. The corresponding result given by Grace, l. c., p. 356, contains a numerical error.

† The construction which follows is due to Mannheim and given by Casey, l. c., p. 210.

‡ We indicate briefly another construction. Lucas has shown that in the sense of least squares, the line passing nearest to the three points A, B, C is the major axis of the ellipse. That line can be constructed by ruler and compass. See Coolidge, American Mathematical Monthly, vol. XX (1912–13), p. 187. Knowing the major axis of the ellipse in position, from any of the sides of the triangle and its mid-point (a tangent to the ellipse and its point of contact) we can construct the major auxiliary circle and hence find the foci.

§ Salmon, Conic Sections, p. 175, Ex. 2.

origin of coördinates. If two of the roots of the polynomial are denoted by α and β, we have to deal with the polynomials

$$f(z) = (z - \alpha)(z - \beta)(z + \alpha + \beta),$$

$$f'(z) = 3z^2 - (\alpha^2 + \alpha\beta + \beta^2).$$

By means of a succession of right triangles, we readily construct

$$\sqrt{(\alpha + \beta)^2 + \alpha^2 + \beta^2},$$

and we easily divide it in the ratio $1 : \sqrt{6}$.

Harvard University,
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THE ASYMPTOTIC EXPANSION OF THE STURM-LIOUVILLE FUNCTIONS.

BY F. H. MURRAY.

1. In the Proceedings of the National Academy of Sciences (vol. 3, pp. 656–659), Professor Birkhoff gave a direct proof of the closure of the set of Sturm-Liouville functions defined by the equation and boundary conditions,

$$\frac{d^2y}{dx^2} + [\rho^2 - g(x)]y = 0, \qquad y(0) = y(1) = 0. \tag{1}$$

This proof was based on a certain asymptotic expansion for these functions, similar to those used by Hobson* and Kneser.† At the suggestion of Professor Birkhoff I have undertaken to develop this expansion in detail, using explicitly the Volterra integral equation of the second kind used more or less implicitly by most writers in this connection; the method of successive approximation employed in the asymptotic development of the characteristic numbers is capable of extension to the functions satisfying the boundary conditions

$$\alpha' y(0) - \alpha y'(0) = 0, \qquad \alpha\alpha' \geqq 0,$$

$$\beta' y(0) + \beta y'(0) = 0, \qquad \beta\beta' \geqq 0.$$

The explicit use of the Volterra integral equation is especially convenient in the study of the differentiability of the characteristic functions with respect to a parameter σ, introduced by replacing $g(x)$ by $\sigma g(x)$.

1. **Some preliminary inequalities.** Assume that $g(x)$ has bounded variation; instead of the system (1) consider first the system

$$\frac{d^2y}{dx^2} + [\rho^2 - \sigma g(x)]y = 0, \qquad y(0) = y(1) = 0. \tag{2}$$

The equation above can be written in the form

$$\frac{d^2y}{dx^2} + \rho^2 y = \sigma g(x)y,$$

which leads to the Volterra integral equation of the second kind,

* Proceedings of the London Math. Soc., ser. (2), vol. 6, p. 374.

† Die Integralgleichungen und ihre Anwendung in der mathematischen Physik, Chap. 3.

$$(3)\qquad y(x) = \alpha \sin \rho(x - \beta) + \frac{\sigma}{\rho}\int_0^x \sin \rho(x - \xi) g(\xi) y(\xi) d\xi.$$

Here α, β are arbitrary constants; if $y(x)$ satisfies the boundary condition $y(0) = 0$, we may assume $\beta = 0$. For convenience assume α positive or zero; ρ, σ are real, and $g(x)$ is real for $0 \leq x \leq 1$. Since $g(x)$ has bounded variation for $0 \leq x \leq 1$, $|g(x)|$ has an upper bound G in this interval.

Suppose

$$K_\rho(x_1\xi) = \sin \rho(x - \xi) g(\xi), \qquad |K_\rho(x_1\xi)| \leq G.$$

Equation (3) becomes,

$$(4)\qquad y(x) = \alpha \sin \rho x + \frac{\sigma}{\rho}\int_0^x K_\rho(x, \xi) y(\xi) d\xi.$$

Suppose

$$(5)\qquad U_\nu = \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} K_\rho(x, \xi_1) K_\rho(\xi_1, \xi_2) \cdots K_\rho(\xi_{\nu-1}, \xi_\nu) \sin \rho\xi_\nu d\xi_1 d\xi_2 \cdots d\xi_\nu.$$

Then the solution of (4) can be given in the form

$$(6)\qquad y(x) = \alpha \left\{ \sin \rho x + \sum_{\nu=1}^{\infty} U_\nu(x) \right\},$$

and this series is dominated by the series

$$(7)\qquad \alpha \left\{ 1 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\frac{\sigma G x}{\rho}\right)^\nu \right\} = \alpha e^{\sigma G x/\rho}.$$

For convenience assume $\alpha = 1$. From (5), (6) it is seen immediately that for large values of $|\rho|$ the first few terms of the series are the important ones. Expand U_1:

$$U_1 = \frac{\sigma}{\rho}\int_0^x \sin \rho(x - \xi_1) \sin \rho\xi_1 g(\xi_1) d\xi_1.$$

Substitute

$$\sin \rho\xi_1 \sin \rho(x - \xi_1) = \tfrac{1}{2}[\cos \rho(x - 2\xi_1) - \cos \rho x],$$

$$U_1 = -\frac{\sigma}{2\rho}\cos \rho x \int_0^x g(\xi_1) d\xi_1 + \frac{\sigma}{2\rho}\int_0^x \cos \rho(x - 2\xi_1) g(\xi_1) d\xi_1.$$

From a lemma by Riemann* it follows that the second term on the right is of the order of $1/\rho^2$.

* Gesammelte Werke, p. 241; Whittaker and Watson, Modern Analysis, 2d ed., p. 166.

Assume

$$h(x) = \int_0^x g(\xi_1)d\xi_1, \qquad h(1) = h.$$

(8)

$$H(x, \sigma, \rho) = \frac{\sigma}{2\rho}\int_0^x \cos \rho(x - 2\xi_1)g(\xi_1)d\xi_1 + \sum_{\nu=2}^{\infty} U_\nu.$$

Then from (6),

(9)
$$y(x) = \sin \rho x - \frac{\sigma h(x)}{2\rho}\cos \rho x + H(x, \sigma, \rho).$$

The function $y(x)$ has already been so determined as to satisfy the first boundary condition $y(0) = 0$, for all values of ρ; it remains to determine the particular values of ρ for which $y(1) = 0$, or the characteristic numbers. This condition becomes:

(10)
$$\tan \rho = \frac{\sigma h}{2\rho} - \frac{H(1, \sigma, \rho)}{\cos \rho} = \phi(\sigma, \rho).$$

This equation can be solved by a method of successive approximations involving a Lipschitz condition of the form

$$| \phi(\sigma, \rho'') - \phi(\sigma, \rho') | < C_\rho | \rho'' - \rho' |.$$

To find an upper bound for C_ρ, compute the partial derivatives $\partial U_\nu/\partial\rho$:

$$\frac{\partial U_\nu}{\partial\rho} = -\frac{\nu}{\rho}U_\nu(x,\sigma, \rho) + \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} \Bigg[\sum_{\kappa=1}^{\nu} K_\rho(x, \xi_1)$$

$$\cdots K_\rho(\xi_{k-2}, \xi_{k-1}, \rho) \frac{\partial}{\partial\rho}K_\rho(\xi_{k-1}, \xi_k)\cdots K_\rho(\xi_{\nu-1}, \xi_\nu) \Bigg] [\sin \rho\xi_\nu d\xi_1 d\xi_2\cdots d\xi_\nu]$$

$$+ \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} K_\rho(x, \xi_1)\cdots K_\rho(\xi_{\nu-1}, \xi_\nu)\xi_\nu \cos \rho\xi_\nu d\xi_1 d\xi_2\cdots d\xi_\nu.$$

Since

$$\frac{\partial}{\partial\rho}K_\rho(\xi_{\kappa-1}, \xi_\kappa) = (\xi_{\kappa-1} - \xi_\kappa) \cos \rho(\xi_{\kappa-1} - \xi_\kappa)g(\xi_\kappa),$$

$$\left| \frac{\partial}{\partial\rho}K_\rho(\xi_{\kappa-1}, \xi_\kappa) \right| \leq G.$$

Consequently

(11)
$$\left| \frac{\partial}{\partial\rho} U_\nu(x, \sigma, \rho) \right| \leq \frac{1}{(\nu - 1)!}\left(\frac{\sigma G x}{\rho}\right)^{\nu-1} \left[\frac{\sigma G x}{\rho}\left(1 + \frac{1}{\rho} + \frac{1}{\nu}\right)\right].$$

From (8),

(12)
$$\frac{\partial H}{\partial\rho} = \frac{-\sigma}{2\rho^2}\int_0^x \cos \rho(x - 2\xi_1)g(\xi_1)d\xi_1$$

$$- \frac{\sigma}{2\rho}\int_0^x (x - 2\xi_1) \sin \rho(x - 2\xi_1)g(\xi_1)d\xi_1 + \sum_{\nu=2}^{\infty}\frac{\partial U_\nu}{\partial\rho}.$$

It follows from (11), (12), and Riemann's lemma that there exists a constant a_1 such that if ρ is real and greater than 1,

(13) $$\left|\frac{\partial H}{\partial \rho}\right| < \frac{a_1}{\rho^2},$$

From (10),

(14) $$\frac{\partial \phi}{\partial \rho} = \frac{-\sigma h}{2\rho^2} - \frac{\cos \rho \dfrac{\partial H}{\partial \rho} + H \sin \rho}{\cos^2 \rho}.$$

From (7), (8), a constant a_2 can be found such that if $\rho > 1$,

(15) $$|H(x, \sigma, \rho)| < \frac{a_2}{\rho^2}.$$

If ρ is so chosen that $|\cos \rho| \geq \frac{1}{2}$, and $|\rho| > 1$, it follows from (13), (14), (15) that for some m,

(16) $$\left|\frac{\partial \phi}{\partial \rho}\right| \leq \frac{m}{\rho^2}.$$

This inequality holds for $0 \leq x \leq 1$, $0 \leq \sigma \leq 1$, and ρ real; it leads immediately to the Lipschitz condition desired.

It will be convenient also to calculate $\partial\phi/\partial\sigma$; from (10),

$$\frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho} = -\frac{\dfrac{\partial H}{\partial \sigma}}{\cos \rho}.$$

If $|\cos \rho| \geq \frac{1}{2}$,

$$\left|\frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho}\right| \leq 2\left|\frac{\partial H}{\partial \sigma}\right|$$

$$\leq 2\left|\frac{1}{2\rho}\int_0^x \cos \rho(x - 2\xi_1) g(\xi_1) d\xi_1 + \sum_{\nu=2}^{\infty} \frac{\partial U_\nu}{\partial \sigma}\right|.$$

Since from (5),

$$\left|\frac{\partial U_\nu}{\partial \sigma}\right| \leq \frac{\nu}{\rho} \cdot \left(\frac{\sigma}{\rho}\right)^{\nu-1} \frac{(Gx)^\nu}{\nu!} \leq \frac{Gx}{\rho}\left(\frac{\sigma G x}{\rho}\right)^{\nu-1} \cdot \frac{1}{(\nu-1)!},$$

we have finally,

(17) $$\left|\frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho}\right| \leq \frac{a_3}{\rho^2},$$

where a_3 is a constant; ρ is real and $|\rho| > 1$, $|\cos \rho| \geq \frac{1}{2}$.

2. **Asymptotic expansion of the characteristic numbers.** In the equation

$$\tan \rho = \phi(\sigma, \rho)$$

put $\rho = \kappa\pi + \epsilon_\kappa$; the equation becomes

(18) $$\tan \epsilon_\kappa = \phi(\sigma, \kappa\pi + \epsilon_\kappa).$$

Let δ_κ be the interval $\kappa\pi - (\pi/3) \leqq \rho \leqq \kappa\pi + (\pi/3)$. From the inequalities already established it is seen immediately that N can be chosen so large that if $\kappa > N$, and if ρ_1, ρ_2 are in the same interval δ_κ, the following inequalities are satisfied:

$$(19) \qquad \begin{aligned} &(a) \quad |\phi(\sigma, \rho_1)| < \frac{C}{(\kappa - 1)\pi}, \\ &(b) \quad |\phi(\sigma, \rho_2) - \phi(\sigma, \rho_1)| < \frac{m|\rho_2 - \rho_1|}{[(\kappa - 1)\pi]^2}, \\ &(c) \quad \frac{4m}{[(\kappa - 1)\pi]^2} < 1, \\ &(d) \quad \frac{2C}{(\kappa - 1)\pi} \cdot \frac{1}{1 - \dfrac{4m}{[(\kappa - 1)\pi]^2}} < \frac{\pi}{6}. \end{aligned}$$

Condition (b) follows from (16) by integration. Let ϵ', ϵ'', $\cdots$, $\epsilon^{(\nu)}$ be defined by the equations

$$(20) \qquad \begin{aligned} \tan \epsilon' &= \phi(\sigma, \kappa\pi), \\ \tan \epsilon'' &= \phi(\sigma, \kappa\pi + \epsilon'), \\ &\cdots\cdots\cdots \\ \tan \epsilon^{(\nu)} &= \phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)}). \end{aligned}$$

In solving this system, observe that if $|\mu| < 1$,

$$(21) \qquad \arctan \mu = \int_0^\mu \frac{dx}{1 + x^2}, \qquad |\arctan \mu| < \frac{|\mu|}{1 - \mu^2}.$$

From this inequality we obtain:

$$|\epsilon'| < \frac{|\phi(\sigma, \kappa\pi)|}{1 - \phi(\sigma, \kappa\pi)^2}.$$

From (19), (a) and (d),

$$|\phi(\sigma, \kappa\pi)| < \frac{\pi}{12}.$$

Consequently

$$|\phi^2(\sigma, \kappa\pi)| < \tfrac{1}{2},$$

$$(22) \qquad |\epsilon'| < 2|\phi(\sigma, \kappa\pi)| < \frac{\pi}{6}.$$

It follows that $(\kappa\pi + \epsilon')$ lies in the interval δ_κ, and ϵ'' can be determined from (20).

Applying the method of induction, we shall assume that $(\kappa\pi + \epsilon')$, $(\kappa\pi + \epsilon'') \cdots (\kappa\pi + \epsilon^{(\nu)})$ lie in the same interval δ_κ; in addition, assume that for $\mu < \nu$,

$$(23) \qquad |\epsilon^{(\mu+1)} - \epsilon^{(\mu)}| < \frac{2C}{(\kappa - 1)\pi}\left[\frac{4m}{[(\kappa - 1)\pi]^2}\right]^{\mu},$$

$$\tan\,[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}] = \frac{\phi(\sigma, \kappa\pi + \epsilon^{(\nu)}) - \phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)})}{1 + \phi(\sigma, \kappa\pi + \epsilon^{(\nu)})\phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)})}.$$

From (19) (*b*), (*d*), and from (23),

$$|\tan\,[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}]| < \frac{2m}{[(\kappa - 1)\pi]^2}\cdot\frac{2C}{(\kappa - 1)\pi}\left[\frac{4m}{[(\kappa - 1)\pi]^2}\right]^{\nu-1}.$$

From (19) (*c*), (*d*), and from (21),

$$[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}] < \frac{2C}{(\kappa - 1)\pi}\left[\frac{4m}{[(\kappa - 1)\pi]^2}\right]^{\nu},$$

$$|\epsilon^{(\nu+1)}| = |\epsilon' + (\epsilon'' - \epsilon') + \cdots + (\epsilon^{(\nu+1)} - \epsilon^{(\nu)})|$$

$$(24) \qquad < \frac{2C}{(\kappa - 1)\pi}\left[1 + \frac{4m}{[(\kappa - 1)\pi]^2} + \cdots + \left[\frac{4m}{[(\kappa - 1)\pi]^2}\right]^{\nu}\right]$$

$$< \frac{\pi}{6},$$

from (19), (*d*). Consequently the inequalities (23), (24) hold for $\nu = 1$, and ϵ''' can be determined from (20), while $(\kappa\pi + \epsilon')$, $(\kappa\pi + \epsilon'')$ lie in δ_κ; $(\kappa\pi + \epsilon''')$ lies in δ_κ, hence $\epsilon^{(4)}$ can be determined, and $(\kappa\pi + \epsilon^{(4)})$ lies in δ_κ, etc. By the principle of induction (23) and (24) hold for $\nu = 1, 2, 3, \cdots, \nu, \cdots$; from (23) the series

$$\epsilon_\kappa = \epsilon' + (\epsilon'' - \epsilon') + (\epsilon''' - \epsilon'') + \cdots$$

converges absolutely and uniformly with respect to σ, for $0 \leq \sigma \leq 1$; consequently ϵ_κ is a continuous function of σ.

To verify that ϵ_κ satisfies (18), observe that in the identity

$$\begin{aligned}\tan \epsilon_\kappa - \phi(\sigma, \kappa\pi + \epsilon_\kappa) &= [\tan \epsilon_\kappa - \tan \epsilon^{(\nu+1)}] \\ &\quad + [\tan \epsilon^{(\nu+1)} - \phi(\sigma, \kappa\pi + \epsilon^{(\nu)})] \\ &\quad + [\phi(\sigma, \kappa\pi + \epsilon^{(\nu)}) - \phi(\sigma, \kappa\pi + \epsilon_\kappa)],\end{aligned}$$

the second term on the right vanishes, while the first and last terms can be shown to approach zero, as ν becomes infinite, with the aid of the inequalities (19) and (23). Since the difference on the left is independent of ν, this difference must vanish.

The asymptotic expansion of ϵ_κ can be determined with the aid of the inequalities established above.

$$\begin{aligned}|\epsilon_\kappa - \epsilon'| &= |(\epsilon'' - \epsilon') + (\epsilon''' - \epsilon'') + \cdots| \\ &< \frac{2C}{(\kappa - 1)\pi}\left[\frac{4m}{[(\kappa-1)\pi]^2} + \cdots\right] \\ &< \frac{\pi}{6}\cdot\frac{4m}{[(\kappa-1)\pi]^2},\end{aligned}$$

$$\begin{aligned}\epsilon' &= \arctan \phi(\sigma, \kappa\pi) \\ &= \phi(\sigma, \kappa\pi) + C_3[\phi(\sigma, \kappa\pi)]^3 \pm \cdots \\ &= \frac{h\sigma}{2\kappa\pi} - \frac{H}{\cos \kappa\pi} + \cdots\end{aligned}$$

$$\begin{aligned}\left|\epsilon_\kappa - \frac{h\sigma}{2\kappa\pi}\right| &= \left|(\epsilon_\kappa - \epsilon') + \left(\epsilon' - \frac{h\sigma}{2\kappa\pi}\right)\right| \\ &< \frac{\alpha}{\kappa^2\pi^2},\end{aligned}$$

$$\epsilon_\kappa = \frac{h\sigma}{2\kappa\pi} + \frac{\overline{H}(\sigma, \kappa)}{\kappa^2\pi^2}, \tag{25}$$

where $\overline{H}$ is a function less than some constant H for all values of $\kappa > N$, and for $0 \leq \sigma \leq 1$. For large values of κ the derivative $d\rho_\kappa/d\sigma$ exists. For suppose σ_1, σ_2 two values of σ in the interval $0 \leq \sigma \leq 1$, ρ_1, ρ_2 the corresponding values of ρ for a given κ.

From the equations

$$\tan \rho_1 = \phi(\sigma_1, \rho_1), \qquad \tan \rho_2 = \phi(\sigma_2, \rho_2),$$

$$\frac{\tan \rho_2 - \tan \rho_1}{\rho_2 - \rho_1} = [1 + \tan \rho_1 \tan \rho_2]\frac{\tan(\rho_2 - \rho_1)}{\rho_2 - \rho_1},$$

we obtain

$$\frac{\rho_2 - \rho_1}{\sigma_2 - \sigma_1} = \frac{\dfrac{\phi(\sigma_2, \rho_1) - \phi(\sigma_1, \rho_1)}{\sigma_2 - \sigma_1}}{[1 + \phi(\sigma_1, \rho_1)\phi(\sigma_2, \rho_2)]\dfrac{\tan(\rho_2 - \rho_1)}{\rho_2 - \rho_1} - \dfrac{\phi(\sigma_2, \rho_2) - \phi(\sigma_2, \rho_1)}{\rho_2 - \rho_1}}.$$

Let the difference $\sigma_2 - \sigma_1$ approach zero; the partial derivatives approached on the right exist and are continuous in ρ, σ; consequently

$$\begin{aligned}\frac{d\rho}{d\sigma} &= \frac{\dfrac{\partial\phi}{\partial\sigma}}{1 + \phi^2 - \dfrac{\partial\phi}{\partial\rho}}, \\ &= \frac{\partial\phi}{\partial\sigma}\left[1 + \frac{\dfrac{\partial\phi}{\partial\rho} - \phi^2}{1 + \phi^2 - \dfrac{\partial\phi}{\partial\rho}}\right].\end{aligned}$$

Consequently from (11),

$$\frac{d\rho}{d\sigma} = \frac{h}{2\rho} + \frac{K_\rho(\sigma)}{\rho^2}, \tag{26}$$

where K_ρ is a function of ρ, σ, bounded for large values of ρ.

3. **The characteristic functions.** In equation (9),

$$y(x) = \sin \rho x - \frac{\sigma h(x)}{2\rho} \cos \rho x + H(x, \sigma, \rho), \tag{9}$$

substitute $\rho = \kappa\pi + \epsilon_\kappa$:

$$\sin \rho x = \sin \kappa\pi x + \epsilon_\kappa x \cos \kappa\pi x + \epsilon_\kappa^2 \psi_1(x, \kappa, \sigma),$$

$$\cos \rho x = \cos \kappa\pi x + \epsilon_\kappa \psi_2(x, \kappa, \sigma),$$

where ψ_1, ψ_2 and hereafter ψ_ν will denote functions of x, κ, σ continuous in x and less than some constant C_ν for $\kappa > N$, and $0 \leq \sigma \leq 1$.

Equation (9) becomes,

$$y(x) = \sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi_3}{\kappa^2}. \tag{9'}$$

Equations (9) and (9′) are identical for $0 \leq \sigma \leq 1$. In (9) compute $\partial y/\partial\sigma$:

$$\begin{aligned} \frac{\partial y}{\partial \sigma} &= \frac{\partial H}{\partial \sigma} - \frac{h(x)}{2\rho} \cos \rho x \\ &\quad + \frac{d\rho}{d\sigma}\left[x \cos \rho x + \frac{\sigma h(x)}{2\rho^2} \cos \rho x + \frac{x\sigma h(x)}{2\rho} \sin \rho x + \frac{\partial H}{\partial \rho}\right] \\ &= \frac{xh - h(x)}{2\kappa\pi} \cos \kappa\pi x + \frac{\psi_4}{\kappa^2}. \end{aligned} \tag{27}$$

From (9′),

$$\frac{\partial y}{\partial \sigma} = \frac{xh - h(x)}{2\kappa\pi} \cos \kappa\pi x + \frac{1}{\kappa^2}\frac{\partial \psi_3}{\partial \sigma}.$$

Comparing results,

$$\frac{\partial \psi_3}{\partial \sigma} = \psi_4.$$

From (9′) compute

$$\int_0^1 y^2(x)dx$$

$$\begin{aligned} \int_0^1 y^2(x)dx &= \int_0^1 \left[\sin^2 \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \sin 2\kappa\pi x + \frac{\psi_5}{\kappa^2}\right] dx \\ &= \frac{1}{2} + \frac{\psi_6}{\kappa^2}, \end{aligned} \tag{28}$$

since from Riemann's Lemma,

$$\left|\int_0^1 \frac{\sigma}{2\kappa\pi}[xh - h(x)] \sin 2\kappa\pi x dx\right| < \frac{c}{\kappa^2}.$$

Also,

$$\frac{d}{d\sigma}\int_0^1 y^2(x)dx = 2\int_0^1 y(x)\left[\frac{1}{2\kappa\pi}(xh - h(x)) \cos \kappa\pi x + \frac{\psi_4}{\kappa^2}\right]dx,$$

applying Riemann's Lemma again, we obtain the result

$$\left|\frac{d}{d\sigma}\int_0^1 y^2(x)dx\right| < \frac{c'}{\kappa^2},$$

where c' as well as c is a constant independent of κ if $\kappa > N$. From (28)

$$(29) \qquad \left|\frac{\partial \psi_6}{\partial \sigma}\right| < c'; \qquad \frac{\partial \psi_6}{\partial \sigma} = \psi_7.$$

Each of the functions $y_\kappa(x)$ is *normalized* by division by the function, of σ,

$$\sqrt{\int_0^1 y_\kappa^2(x)dx}.$$

$$\frac{1}{\int_0^1 y_\kappa^2(x)dx} = \frac{1}{\frac{1}{2} + \frac{\psi_6}{\kappa^2}} = 2 - \frac{\psi_6}{\kappa^2\left(\frac{1}{2} + \frac{\psi_6}{\kappa^2}\right)} = 2 + \frac{\psi_8}{\kappa^2},$$

$$\sqrt{2 + \frac{\psi_8}{\kappa^2}} = \sqrt{2} + \sqrt{2}\left[\sqrt{1 + \frac{\psi_8}{2\kappa^2}} - 1\right].$$

If $|z| < 1$,

$$\sqrt{1 + z} - 1 = \frac{z}{2} - \frac{1\cdot 1}{2\cdot 4}z^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}z^3 \pm \cdots,$$

$$|\sqrt{1 + z} - 1| < \frac{|z|}{2}[1 + |z| + |z|^2 + \cdots]$$

$$< \frac{1}{2}\frac{|z|}{1 - |z|}.$$

If the constant N has been chosen sufficiently large,

$$\left|\frac{\psi_6}{2\kappa^2}\right| < \frac{1}{2},$$

and accordingly

$$\sqrt{2 + \frac{\psi_8}{\kappa^2}} = \sqrt{2} + \frac{\psi_9}{\kappa^2}.$$

$$
(30)\qquad \begin{aligned} \bar{y}(x) &= \frac{y(x)}{\sqrt{\int_0^1 y^2(x)dx}} = y(x)\left[\sqrt{2} + \frac{\psi_9}{\kappa^2}\right] \\ &= \sqrt{2}\left\{\sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi_{10}}{\kappa^2}\right\}. \end{aligned}
$$

$$
(31)\qquad \begin{aligned} \frac{\partial \bar{y}(x)}{\partial \sigma} &= \frac{\sqrt{\int_0^1 y^2(x)dx}\,\frac{\partial y(x)}{\partial \sigma} - \frac{y(x)}{\sqrt{\int_0^1 y^2(x)dx}}\int_0^1 y(x)\,\frac{\partial y(x)}{\partial \sigma}\,dx}{\int_0^1 y^2(x)dx} \\ &= \sqrt{2}\left\{\frac{xh - h(x)}{2\kappa\pi} \cos \kappa\pi x + \frac{\psi_{11}}{\kappa^2}\right\}. \end{aligned}
$$

Comparing (30), (31), we obtain the result:

$$
(32)\qquad \frac{\partial \psi_{10}}{\partial \sigma} = \psi_{11}.
$$

Hence, finally, the normalized functions $y_\kappa(x)$ satisfying the system (2) are given asymptotically in the form

$$
y_\kappa(x) = \sqrt{2}\left\{\sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi(x, \sigma, \kappa)}{\kappa^2}\right\},
$$

$$
(32)\qquad \left|\frac{\partial \psi(x, \sigma, \kappa)}{\partial \sigma}\right| < K,
$$

where K is independent of κ, $\sigma < 1$ for $\kappa > N$.

4. **Nature of the solutions for small values of ρ.** In the preceding discussion the parameter ρ was assumed larger than a constant N'; to complete the discussion it is necessary to study the functions $y(x)$ for small values of ρ. For this purpose write the equation of (2) in the form

$$
(2')\qquad \frac{d^2y}{dx^2} = [\sigma g(x) - \lambda]y.
$$

The solutions of the homogeneous equation obtained by equating the left-hand member to zero are 1 and x; the corresponding integral equation becomes,

$$
y(x) = \alpha + \beta x + \int_0^x (x - \xi)[\sigma g(\xi) - \lambda]y(\xi)d\xi,
$$

since $y(0) = 0$, $\alpha = 0$; for convenience assume $\beta = 1$,

$$
y(x) = x + \int_0^x (x - \xi)[\sigma g(\xi) - \lambda]y(\xi)d\xi.
$$

For values of λ less than or equal to N'^2 the kernel is bounded, and continuous and differentiable with respect to σ, λ; the series expansion of the solution of this integral equation converges uniformly as before for $0 \leq \sigma \leq 1$, $|\lambda| \leq N'^2$. Consequently the solution $y(x)$ is continuous in λ, σ, and as before the partial derivatives with respect to these parameters can be computed from the series expansion.

If $\partial y/\partial\lambda$ has continuous first and second partial derivatives with respect to x, such that

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}\left(\frac{\partial y}{\partial x}\right),$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}\left(\frac{\partial^2 y}{\partial x^2}\right),$$

then from the equation

$$(2') \qquad y'' + (\lambda - \sigma g(x))y = 0,$$

we obtain:

$$(33) \qquad \frac{d^2}{dx^2}\left(\frac{\partial y}{\partial\lambda}\right) + (\lambda - \sigma g(x))\frac{\partial y}{\partial\lambda} = -y.$$

Let $y(x)$ be a solution of (2') vanishing for $x = 0$, and suppose $\bar{y}(x)$ any solution not vanishing at the origin for any value of σ. The solution of (33) becomes,

$$\frac{\partial y}{\partial\lambda} = c_1 y(x) + c_2\bar{y}(x) + \frac{1}{y\bar{y}' - \bar{y}y'}\int_0^x [\bar{y}(x)y(\xi) - y(x)\bar{y}(\xi)][-y(\xi)]d\xi.$$

When $x = 0$, $\partial y/\partial\lambda = 0$, $y(0) = 0$, hence $c_2 = 0$.

$$\frac{\partial y}{\partial\lambda} = c_1 y(x) - \frac{1}{y\bar{y}' - \bar{y}y'}\int_0^x [\bar{y}(x)y(\xi) - y(x)\bar{y}(\xi)]y(\xi)d\xi.$$

The conditions $\partial^2 y/\partial x\partial\lambda = \partial^2 y/\partial\lambda\partial x$, etc., are easily seen to be satisfied. Suppose λ a characteristic number; then $y(1) = 0$. Put $x = 1$.

$$\left[\frac{\partial y}{\partial\lambda}\right]_{x=1} = \frac{-\bar{y}(1)}{y\bar{y}' - \bar{y}y'}\int_0^1 y^2(\xi)d\xi.$$

Since $y(1) = 0$, $\bar{y}(1) \neq 0$, hence

$$\left[\frac{\partial y}{\partial\lambda}\right]_{x=1} \neq 0.$$

It follows immediately that $d\lambda/d\sigma$ can be computed from the equation

$$\left[\frac{\partial y}{\partial\sigma}\right]_{x=1} + \left[\frac{\partial y}{\partial\lambda}\frac{d\lambda}{d\sigma}\right]_{x=1} = 0$$

and is finite for $\rho < N'$.

Consequently if σ_1, σ_2 lie in the interval $0 \leqq \sigma \leqq 1$, $\sigma_1 < \sigma_2$,

$$|y_\kappa(x, \sigma_2) - y_\kappa(x, \sigma_1)| \leqq \int_{\sigma_1}^{\sigma_2} \left| \frac{\partial y_\kappa}{\partial \sigma} + \frac{\partial y_\kappa}{\partial \lambda} \frac{d\lambda}{d\sigma} \right| d\sigma$$

$$< A_\kappa |\sigma_2 - \sigma_1|,$$

where A_κ is independent of σ, for $0 \leqq \sigma \leqq 1$.

Since N' is finite, some A_κ is equal to or greater than any of the others; call this one A:

$$(34) \qquad |y_\kappa(x, \sigma_2) - y_\kappa(x, \sigma_1)| < A |\sigma_2 - \sigma_1|$$

$$\rho_\kappa < N', \qquad 0 \leqq \sigma_1 < \sigma_2 \leqq 1.$$

It has been seen that $y_\kappa(x, \sigma)$ is continuous in σ; it follows that $y_\kappa(x, \sigma)$ vanishes just $(\kappa - 1)$ times in the interval $0 < x < 1$. For this is true when $\sigma = 0$; as σ increases from 0 to 1 the end-points of the curve $y = y_\kappa(x)$ remain fixed, and consequently zeros can enter or disappear only if for at least one value of σ the curve becomes tangent to the x-axis. At the point of tangency $y = y' = 0$; since $y(x)$ satisfies the differential equation (2) $y(x)$ vanishes identically. But from (32) this is seen to be impossible for very large values of κ; suppose N so large that for $\kappa \geqq N$ $y_\kappa(x, \sigma)$ does not vanish identically for any value of σ between 0 and 1. Then $y_\kappa(x, \sigma)$ has the same number of zeros for $\sigma = 1$ as for $\sigma = 0$, or $(\kappa - 1)$ zeros. Now for $\kappa \leqq N$ the characteristic functions can be so ordered that $y_\kappa(x, 1)$ has one more zero than $y_{\kappa-1}(x, 1)$; assuming the functions y_κ in this order, y_{N-1} has $(N - 2)$ zeros, y_{N-2} has $(N - 3)$ zeros,—hence for $\kappa \leqq N$ as well as for $\kappa > N$, y_κ has just $(\kappa - 1)$ zeros in the interval

$$0 < x < 1.$$

The equation and inequality (32), together with (34), are sufficient for an immediate application of the theorem of Professor Birkhoff, with the aid of which the closure of the set of normalized functions satisfying (1) is established.

ON THE CONFORMAL MAPPING OF A REGION INTO A PART OF ITSELF.

By J. F. Ritt.

1. This note will extend to domains of any degree of connectivity a theorem proved for simply connected domains by G. Julia in the preliminaries to his prize memoir on the iteration of rational functions.*

Consider, in the plane of the variable z, a closed and bounded domain Δ, consisting of a two-dimensional continuum plus its boundary. Let a function $f(z)$ be analytic in Δ and assume throughout Δ values which correspond to inner points of Δ. Then $f(z)$ maps Δ conformally on a Riemann surface of one or more sheets, every interior and boundary point of which lies within Δ. If we project the points of this Riemann surface upon the z plane, we obtain a closed domain Δ_1. To every value assumed by $f(z)$ in Δ, no matter how many times, there corresponds one and only one point of Δ_1. The domain Δ_1 consists of inner points and boundary points, each boundary point being a limit point of inner points. Every value which $f(z)$ takes at an inner point of Δ gives an inner point of Δ_1; the boundary points of Δ_1 correspond only to values which $f(z)$ takes at boundary points, but not at inner points, of Δ; in certain cases, however, values which $f(z)$ takes only on the boundary of Δ may give inner points of Δ_1. It is easy to show that the inner points of Δ_1 form a continuum, but we shall not have occasion to use this fact.

It is evident that $f(z)$ transforms Δ_1 into a closed domain Δ_2 whose points are all inner points of Δ_1, Δ_2 into a smaller domain Δ_3, etc.

The theorem we are to prove states that:

(a) *In the transformation of Δ into Δ_1, one and only one point of Δ stays fixed.*

(b) *At this fixed point a, we have* $|f'(a)| < 1$.

(c) *The domains $\Delta, \Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$ converge to the fixed point.*

The application of this theorem to doubly connected regions shows that it is impossible to shrink a ring conformally into a ring situated in its interior; that is, if the given ring is bounded by certain curves C_1 and C_2 (C_2 interior to C_1), it is impossible to map the ring conformally in a one-to-one manner upon a ring bounded by curves Γ_1 and Γ_2, with Γ_1 interior to C_1 and C_2 interior to Γ_2. For it is clear that the points of

* G. Julia, Sur l'itération des fonctions rationnelles, Journ. de Math., 1918, p. 69.

the second ring would be mapped on a third ring, interior to the second, the third ring on a fourth in its interior, etc., and that the sequence of rings thus obtained could not converge to a point.

It will be seen that Julia's proof, which consists in mapping Δ on a circle, in showing the existence of the fixed point by the theory of the zeros of analytic functions and in using Schwarz's lemma for the second and third items of the theorem, cannot be used when Δ is multiply connected. We shall handle the general case by means of Montel's normal families of functions, which Julia uses later himself, to the greatest advantage, in his remarkable paper.

2. We denote the nth iterate of $f(z)$ by $f_n(z)$. Since, for every z in Δ, $f(z)$ also lies in Δ, it is clear that all of the iterates

$$(1) \qquad f_1(z),\ f_2(z),\ \cdots,\ f_n(z),\ \cdots$$

are defined throughout Δ, and have a common upper bound for their moduli in Δ; they constitute a normal family in the sense of Montel.

We can therefore select from the sequence (1) a new sequence

$$(2) \qquad f_{i_1}(z),\ f_{i_2}(z),\ \cdots,\ f_{i_n}(z),\ \cdots$$

which converges uniformly in every closed domain interior to Δ to an analytic function $\varphi(z)$.*

The central part of the proof consists in showing that $\varphi(z)$ is a constant. Let us grant this fact for the moment; we shall see that the theorem follows directly from it.

The functions of the sequence (2) converge uniformly in Δ_1, which is a closed domain interior to Δ. The values assumed by $f_{i_n}(z)$ in Δ_1 are the affixes of the points of Δ_{i_n+1}. Hence if $\varphi(z)$ is a constant a, the domains

$$\Delta_{i_1+1},\ \Delta_{i_2+1},\ \cdots,\ \Delta_{i_n+1},\ \cdots$$

must converge to the point a. But since every Δ_n contains Δ_{n+1}, it is clear that the domains of the entire sequence

$$\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n,\ \cdots$$

converge to a.

The point a must stay fixed in the mapping, otherwise, for large values of n, the domain Δ_n which is very small, and close to a, could not contain Δ_{n+1}, which would be very small and close to $f(a)$. The fixed point a is clearly interior to every Δ_n.

Finally we must have $|f'(a)| < 1$. For since $f(a) = a$, we have

$$f_{i_n}'(a) = f'[f_{i_n-1}(a)]\cdot f'[f_{i_n-2}(a)]\cdots f'(a) = [f'(a)]^{i_n}.$$

* See, for instance, Montel, Les Séries de Polynomes, p. 22.

But

$$\varphi'(a) = \lim_{n \to \infty} f_{i_n}'(a),$$

and since $\varphi'(z)$ is identically zero, we must have

$$|f'(a)| < 1.$$

It remains now only to prove that $\varphi(z)$ is a constant.

We shall show first that if $\varphi(z)$ is not a constant, the points corresponding to the values assumed by $\varphi(z)$ in any one of the domains Δ_p ($p = 1, 2, \cdots$),* are the points common to $\Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$.†

Firstly, if ω is interior to each of these domains, it is interior, in particular, to the domains of the sequence

$$\Delta_{i_1+p}, \ \Delta_{i_2+p}, \ \cdots, \ \Delta_{i_n+p}, \ \cdots.$$

Since $f_{i_n}(z)$ transforms Δ_p into Δ_{i_n+p}, each function of the sequence (2) assumes the value ω at some point of Δ_p. It is the simplest matter to show from this that $\varphi(z)$ also assumes the value ω in Δ_p.

Conversely, let $\varphi(z)$ assume the value ω at some point ζ of Δ_p. If $\varphi(z)$ is not a constant, there must exist a circumference lying wholly within Δ_{p-1}, with center at ζ, on no point of which $\varphi(z)$ is zero; there is a positive M such that, along this circumference,

$$|\varphi(z) - \omega| > M.$$

For n sufficiently great, we have, along the same circumference,

$$|f_{i_n}(z) - \varphi(z)| < M.$$

Hence, by a well-known theorem on the zeros of analytic functions, the function

$$f_{i_n}(z) - \omega = [\varphi(z) - \omega] + [f_{i_n}(z) - \varphi(z)]$$

has the same number of zeros within the circumference as $\varphi(z) - \omega$. What is the same, ω is interior to Δ_{i_n+p-1} for n sufficiently large, and hence it is interior to every Δ_n.

Thus $\varphi(z)$ transforms every Δ_p into the same domain δ which consists of the points common to all of the domains $\Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$.

In transforming Δ_p into δ, the boundary points of δ come only from the boundary points of Δ_p; in transforming Δ_{p+1} into δ, the boundary

* We take p at least equal to 1, because we have not shown that the sequence (2) converges in a domain which includes Δ.

† The proof of this fact will be recognized as a familiar argument in the theory of normal families.

points of δ must come from the boundary points of Δ_{p+1}. But since the boundary points of Δ_{p+1} are inner points of Δ_p, the boundary points of Δ_{p+1} can give only inner points of δ.

The assumption that $\varphi(z)$ is not a constant has forced a contradiction. The theorem is proved.

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CONJUGATE NETS R AND THEIR TRANSFORMATIONS.*

By Luther Pfahler Eisenhart.

1. A rectilinear congruence for which the asymptotic lines on the two focal surfaces correspond is called a W-congruence. When the tangents to the curves of each family of a conjugate system of curves on a surface form W congruences, the system is called a net R.* It is the purpose of this paper to establish two types of transformations of an R net into R nets, called *transformations* W and *transformations* T.

If N is an R net, each pair of solutions of two completely integrable partial differential equations of the second order determine a W transform $\bar{N}$ which is an R net; the nets N and $\bar{N}$ are on the focal surfaces of a W-congruence, and either net is a *derived net* of the other, in the sense of Guichard. These transformations W admit a theorem of permutability, that is if $\bar{N}_1$ and $\bar{N}_2$ are W transforms of N, there exists an R net N_{12} which is a W transform of $\bar{N}_1$ and $\bar{N}_2$.

In a previous paper‡ the author established a theory of *transformations* T of any net whatsoever such that if N_1 is a T transform of a net N, the developables of the congruence of lines joining corresponding points of N and N_1 meet the two surfaces on which N and N_1 lie in these nets. In § 8 it is shown that an R net admits a group of transformations T into R nets. Moreover, these transformations admit theorems of permutability similar to W transformations.

In § 10 it is shown that if $\bar{N}$ and N_1 are W and T transforms respectively of an R net N, there exists a net $\bar{N}_1$ which is a T transform of $\bar{N}$ and a W transform of N_1.

In another paper§ we apply these results to the surfaces applicable to a quadric and show that the transformations B_k of these surfaces established by Bianchi‖ are of the type W, and that the transformations of Guichard¶ are of the type T.

2. **Differential equations of a net.** If x, y, z are the cartesian coördinates

* Presented to the International Congress of Mathematicians at Strasbourg, Sept. 28, 1920.

† Tzitzeica, Comptes Rendus, vol. 152 (1911), p. 1077; also Demoulin, Comptes Rendus, vol. 153 (1912), p. 590.

‡ Transactions of the American Mathematical Society, vol. 18 (1917), pp. 99–124. This paper will be referred to as M.

§ Proceedings of the Strasbourg Congress.

‖ Lezioni, vol. 3.

¶ Mémoires á L'Academie des Sciences, vol. 34 (1909).

of a surface S upon which the parametric curves form a conjugate system or net, the coördinates are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial\theta}{\partial v}, \tag{1}$$

which we call the point equation of the net N. This follows from one of the three *equations of Gauss* for S.* From the other two of these equations it follows that x, y and z are solutions also of an equation of the form

$$\frac{\partial^2\theta}{\partial v^2} = r\frac{\partial^2 \theta}{\partial u^2} + a'\frac{\partial \theta}{\partial u} + b'\frac{\partial\theta}{\partial v}, \tag{2}$$

where

$$r = \frac{D''}{D}, \tag{3}$$

the functions D and D'' being second fundamental coefficients of S.

In order that two equations of the form (1) and (2) admit three independent solutions, it is necessary and sufficient that the functions a, b, a', b' and r satisfy three differential equations of condition. Instead of calculating these conditions, we determine the conditions that the more general system

$$\begin{aligned} \frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial\theta}{\partial v} + c\theta, \\ \frac{\partial^2\theta}{\partial v^2} &= r\frac{\partial^2 \theta}{\partial u^2} + a'\frac{\partial \theta}{\partial u} + b'\frac{\partial\theta}{\partial v} + c'\theta \end{aligned} \tag{4}$$

admit four independent solutions.

Equating the expressions for $\partial^3\theta/\partial u\partial v^2$ obtained from these equations by differentiation, we have

$$\frac{\partial^3 \theta}{\partial u^3} = A_1\frac{\partial^2 \theta}{\partial u^2} + B_1\frac{\partial \theta}{\partial u} + C_1\frac{\partial\theta}{\partial v} + D_1\theta, \tag{5}$$

where

$$\begin{aligned} A_1 &= \frac{\partial}{\partial u}\log\frac{b}{r} - \frac{a'}{r}, \\ B_1 &= \frac{1}{r}\left(\frac{1}{a}\frac{\partial^2 a}{\partial v^2} + a'\frac{\partial}{\partial u}\log\frac{b}{a'} - b'\frac{\partial \log a}{\partial v} - c'\right), \\ C_1 &= \frac{1}{r}\left(\frac{\partial^2 \log b}{\partial u\partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c - \frac{\partial b'}{\partial u}\right), \\ D_1 &= \frac{1}{r}\left(c\frac{\partial \log a}{\partial v} + c'\frac{\partial \log b}{\partial u} + \frac{\partial c}{\partial v} - cb' - \frac{\partial c'}{\partial u}\right). \end{aligned} \tag{6}$$

* Cf. Eisenhart, Differential Geometry, p. 154. Hereafter a reference to this book will be written, E., p. 154.

Differentiating the first of (4) with respect to u, we obtain

$$(7) \qquad \frac{\partial^3\theta}{\partial u^2\partial v} = A_2\frac{\partial^2\theta}{\partial u^2} + B_2\frac{\partial\theta}{\partial u} + C_2\frac{\partial\theta}{\partial v} + D_2\theta,$$

where

$$(8) \qquad \begin{aligned} A_2 &= \frac{\partial \log a}{\partial v}, & B_2 &= \frac{\partial^2 \log a}{\partial u\partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \\ C_2 &= \frac{1}{b}\frac{\partial^2 b}{\partial u^2}, & D_2 &= c\frac{\partial}{\partial u}\log bc. \end{aligned}$$

When from (5) and (7) we express the condition

$$\frac{\partial}{\partial v}\left(\frac{\partial^3\theta}{\partial u^3}\right) = \frac{\partial}{\partial u}\left(\frac{\partial^3\theta}{\partial u^2\partial v}\right),$$

we get an equation of the form

$$(9) \qquad P\frac{\partial^2\theta}{\partial u^2} + Q\frac{\partial\theta}{\partial u} + R\frac{\partial\theta}{\partial v} + S\theta = 0,$$

where P, Q, R and S are determinate functions. If we do not have

$$(10) \qquad P = Q = R = S = 0,$$

equations (4) and (9) admit at most three independent solutions. Hence (10) must hold. When their expressions are calculated we find that (10) is equivalent to

$$(11) \qquad \begin{aligned} &\frac{\partial A_1}{\partial v} + C_1 r = \frac{\partial A_2}{\partial u} + B_2, \\ &\frac{\partial B_1}{\partial v} + A_1B_2 + a'C_1 = \frac{\partial B_2}{\partial u} + C_2\frac{\partial \log a}{\partial v} + D_2, \\ &\frac{\partial C_1}{\partial v} + A_1C_2 + B_1\frac{\partial \log b}{\partial u} + b'C_1 + D_1 = \frac{\partial C_2}{\partial u} + A_2C_1 + C_2\frac{\partial \log b}{\partial u}, \\ &\frac{\partial D_1}{\partial v} + A_1D_2 + B_1c + C_1c' = \frac{\partial D_2}{\partial u} + A_2D_1 + C_2c. \end{aligned}$$

It is readily seen that when these conditions are satisfied, equations (4), (5) and (7) are consistent, and consequently equations (4) admit four independent solutions.

When we take equations (1) and (2) in place of (4), we have $c = c' = D_1 = D_2 = 0$, and the last of (11) is satisfied identically.

3. **Equations of a net** R. When the tangents are drawn to the curves $v =$ const., the second focal net of the congruence of tangents is given by

$$x_{-1} = x - \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial u}.* \tag{12}$$

By differentiation we have

$$\begin{aligned} \frac{\partial x_{-1}}{\partial u} &= \frac{C_2}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x}{\partial u} - \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial^2 x}{\partial u^2}, \\ \frac{\partial x_{-1}}{\partial v} &= -K \frac{1}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x}{\partial u}, \end{aligned} \tag{13}$$

where

$$K = -\frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial \log a}{\partial v} \frac{\partial \log b}{\partial u}. \tag{14}$$

With the aid of (5) we find

$$\begin{aligned} \frac{\partial^2 x_{-1}}{\partial u^2} &= \left(A_1 - \frac{C_2}{\frac{\partial \log b}{\partial u}} - \frac{\frac{\partial^2 \log b}{\partial u^2}}{\frac{\partial \log b}{\partial u}} \right) \frac{\partial x_{-1}}{\partial u} \\ &\quad + \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial u} \left[\frac{\partial}{\partial u} \left(\frac{C_2}{\frac{\partial \log b}{\partial u}} \right) - B_1 + \left(\frac{C_2}{\frac{\partial \log b}{\partial u}} - A_1 \right) \frac{C_2}{\frac{\partial \log b}{\partial u}} \right] \\ &\quad - \frac{C_1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial v}, \\ \frac{\partial^2 x_{-1}}{\partial v^2} &= \frac{\partial}{\partial v} \log \frac{K}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x_{-1}}{\partial v} - \frac{K}{\left(\frac{\partial \log b}{\partial u}\right)^2} \\ &\quad \times \left(\frac{\partial \log a}{\partial v} \frac{\partial x}{\partial u} + \frac{\partial \log b}{\partial u} \frac{\partial x}{\partial v} \right). \end{aligned} \tag{15}$$

From (3) it follows that if the asymptotic lines are to correspond on the two focal surfaces, we must have an equation of the form

$$\frac{\partial^2 x_{-1}}{\partial v^2} = r \frac{\partial^2 x_{-1}}{\partial u^2} + a_{-1}' \frac{\partial x_{-1}}{\partial u} + b_{-1}' \frac{\partial x_{-1}}{\partial v}. \tag{16}$$

---

* E., p. 405.

From (13) and (16) it follows that a necessary and sufficient condition is that $C_1 r = K$, or in consequence of (6),

$$\frac{\partial b'}{\partial u} = 2\frac{\partial^2 \log b}{\partial u \partial v}. \tag{17}$$

In like manner the condition that the tangents to the curves $u = \text{const.}$ of N from a W congruence is

$$\frac{\partial a'}{\partial v} = 2\frac{\partial^2 \log a}{\partial u \partial v}. \tag{18}$$

Hence equations (17) and (18) constitute the condition that N be an R net. When these conditions are satisfied we have from the first of (11) that $\partial^2 \log r/\partial u \partial v = 0$. Consequently we have the theorem of Tzitzeica:

An R net is isothermal conjugate.

Accordingly the parameters u and v can be chosen so that $r = -1$. Since a and b in (1) are determined only to within factors which are functions of u and v respectively, we have the theorem:

The two differential equations satisfied by the cartesian coördinates of an R net are reducible to the forms

$$\begin{gathered} \frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial \theta}{\partial v}, \\ \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = 2\frac{\partial \log a}{\partial u}\frac{\partial \theta}{\partial u} + 2\frac{\partial \log b}{\partial v}\frac{\partial \theta}{\partial v}. \end{gathered} \tag{19}$$

The complete determination of R nets requires the solution of the two equations to be satisfied by a and b which follow from the second and third of (11).

4. **Derived nets.** If N is a net whose point equation is (1), the equations

$$\begin{gathered} \frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \quad \frac{\partial x'}{\partial v} = l\frac{\partial x}{\partial v}; \quad \frac{\partial y'}{\partial u} = h\frac{\partial y}{\partial u}, \quad \frac{\partial y'}{\partial v} = l\frac{\partial y}{\partial v}, \\ \frac{\partial z'}{\partial u} = h\frac{\partial z}{\partial u}, \quad \frac{\partial z'}{\partial v} = l\frac{\partial z}{\partial v} \end{gathered} \tag{20}$$

are consistent provided h and l satisfy the equations

$$\frac{\partial h}{\partial v} = (l - h)\frac{\partial \log a}{\partial v}, \qquad \frac{\partial l}{\partial u} = (h - l)\frac{\partial \log b}{\partial u}. \tag{21}$$

It is readily found that x', y', z' satisfy the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v}\log ah\frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u}\log bl\frac{\partial \theta}{\partial v}, \tag{22}$$

and consequently are the cartesian coördinates of a net N' *parallel* to N, as follows from the form of (20). Moreover, whenever the cartesian coördinates of two surfaces satisfy equations of the form (20), the tangents to the parametric curves at corresponding points are parallel, and these curves form nets on the two surfaces.

When the lines of a congruence H lie in the tangent planes of a net N, the developables of H correspond to the curves of N and the focal points of H lie on the tangents to the curves of N, the congruence H is said to be *harmonic* to N. Each solution θ of (1) determines such a harmonic congruence. The coördinates of the foci are of the forms

$$x - \frac{\theta}{\frac{\partial \theta}{\partial u}} \frac{\partial x}{\partial u}, \qquad x - \frac{\theta}{\frac{\partial \theta}{\partial v}} \frac{\partial x}{\partial v}. * \tag{23}$$

If H_1 and H_2 are two congruences harmonic to N determined by solutions θ_1 and θ_2 of (1), corresponding lines of H_1 and H_2 meet in a point $\bar{M}$ whose coördinates are of the form

$$\bar{x} = x + p\frac{\partial x}{\partial u} + q\frac{\partial x}{\partial v}, \tag{24}$$

where

$$\begin{gathered} p = \frac{1}{\Delta}\left(\theta_1 \frac{\partial \theta_2}{\partial v} - \theta_2 \frac{\partial \theta_1}{\partial v}\right), \qquad q = \frac{1}{\Delta}\left(\theta_2 \frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial \theta_2}{\partial u}\right), \\ \Delta = \frac{\partial \theta_2}{\partial u}\frac{\partial \theta_1}{\partial v} - \frac{\partial \theta_2}{\partial v}\frac{\partial \theta_1}{\partial u}. \end{gathered} \tag{25}$$

By differentiation we have

$$\begin{aligned} \frac{\partial \bar{x}}{\partial u} &= p\left[\frac{\partial^2 x}{\partial u^2} + \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial v} - \frac{\partial^2 \theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial v}\right)\frac{\partial x}{\partial u}\right. \\ &\qquad \left. - \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial u} - \frac{\partial^2 \theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial u}\right)\frac{\partial x}{\partial v}\right], \\ \frac{\partial \bar{x}}{\partial v} &= q\left[\frac{\partial^2 x}{\partial v^2} + \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial v} - \frac{\partial^2 \theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial v}\right)\frac{\partial x}{\partial u}\right. \\ &\qquad \left. - \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial u} - \frac{\partial^2 \theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial u}\right)\frac{\partial x}{\partial v}\right]. \end{aligned} \tag{26}$$

It is readily seen that the functions θ_1' and θ_2' defined by

$$\frac{\partial \theta_1'}{\partial u} = h\frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \theta_1'}{\partial v} = l\frac{\partial \theta_1}{\partial v}; \quad \frac{\partial \theta_2'}{\partial u} = h\frac{\partial \theta_2}{\partial u}, \quad \frac{\partial \theta_2'}{\partial v} = l\frac{\partial \theta_2}{\partial v} \tag{27}$$

are solutions of (22), the point equation of the net N' parallel to N as

* Guichard, Annales de L'Ecole Normale, Ser. 3, vol. 14 (1897).

given by (20). We call θ_1 and θ_1' *corresponding solutions* of (1) and (22); likewise θ_2 and θ_2'.

By means of θ_1' and θ_2' we obtain congruences H_1' and H_2' harmonic to N', and corresponding lines meet in $\bar{M}'$ whose coördinates are

$$(28) \qquad \bar{x}' = x' + p'\frac{\partial x'}{\partial u} + q'\frac{\partial x'}{\partial v},$$

where p' and q' are analogous to (25) in θ_1' and θ_2'. The derivatives of $\bar{x}'$ are expressible by means of the preceding formulas in the forms

$$(29) \qquad \frac{\partial \bar{x}'}{\partial u} = \frac{\theta_2'\dfrac{\partial\theta_1}{\partial v} - \theta_1'\dfrac{\partial\theta_2}{\partial v}}{\theta_2\dfrac{\partial\theta_1}{\partial v} - \theta_1\dfrac{\partial\theta_2}{\partial v}}\frac{\partial\bar{x}}{\partial u}, \qquad \frac{\partial \bar{x}'}{\partial v} = \frac{\theta_2'\dfrac{\partial\theta_1}{\partial u} - \theta_1'\dfrac{\partial\theta_2}{\partial u}}{\theta_2\dfrac{\partial\theta_1}{\partial u} - \theta_1\dfrac{\partial\theta_2}{\partial u}}\frac{\partial\bar{x}}{\partial v}.$$

Since these equations are of the form (20), the points $\bar{M}$ and $\bar{M}'$ describe nets, $\bar{N}$ and $\bar{N}'$, which are parallel to one another. Guichard* calls $\bar{N}$ a derived net of N.

Since θ_1 and θ_2 are solutions of (1), we have

$$(30) \qquad \begin{aligned}
\frac{\partial p}{\partial u} &= -1 - q\frac{\partial \log a}{\partial v} + \frac{p}{\Delta}\left(\frac{\partial^2\theta_1}{\partial u^2}\frac{\partial\theta_2}{\partial v} - \frac{\partial^2\theta_2}{\partial u^2}\frac{\partial\theta_1}{\partial v}\right),\\
\frac{\partial p}{\partial v} &= -p\frac{\partial \log a}{\partial v} + \frac{1}{\Delta}\left(\theta_1\frac{\partial^2\theta_2}{\partial v^2} - \theta_2\frac{\partial^2\theta_1}{\partial v^2}\right) - \frac{p}{\Delta}\left(\frac{\partial^2\theta_1}{\partial v^2}\frac{\partial\theta_2}{\partial u} - \frac{\partial^2\theta_2}{\partial v^2}\frac{\partial\theta_1}{\partial u}\right),\\
\frac{\partial q}{\partial u} &= -q\frac{\partial \log b}{\partial u} + \frac{1}{\Delta}\left(\theta_2\frac{\partial^2\theta_1}{\partial u^2} - \theta_1\frac{\partial^2\theta_2}{\partial u^2}\right) - \frac{q}{\Delta}\left(\frac{\partial^2\theta_2}{\partial u^2}\frac{\partial\theta_1}{\partial v} - \frac{\partial^2\theta_1}{\partial u^2}\frac{\partial\theta_2}{\partial v}\right),\\
\frac{\partial q}{\partial v} &= -1 - p\frac{\partial \log b}{\partial u} + \frac{q}{\Delta}\left(\frac{\partial^2\theta_2}{\partial v^2}\frac{\partial\theta_1}{\partial u} - \frac{\partial^2\theta_1}{\partial v^2}\frac{\partial\theta_2}{\partial u}\right).
\end{aligned}$$

With the aid of these expressions we show that the point equation of $\bar{N}$ is

$$(31) \qquad \frac{\partial^2\bar{\theta}}{\partial u\partial v} = \frac{\partial}{\partial v}\log ap\frac{\partial\bar{\theta}}{\partial u} + \frac{\partial}{\partial u}\log bq\frac{\partial\bar{\theta}}{\partial v}.$$

5. **Reciprocally derived nets.** Ordinarily N is not a derived net of $\bar{N}$. When it is, we say that N is a *reciprocally derived* net. Tzitzeica† was the first to consider nets of this kind. If N is a derived net of $\bar{N}$, the tangent planes of $\bar{N}$ pass through corresponding points of N just as the tangent planes of N pass through the corresponding points of $\bar{N}$. Hence the surfaces S and $\bar{S}$ on which N and $\bar{N}$ lie are the focal surfaces of the congruence $\bar{G}$ of lines joining corresponding points. Since the lines of $\bar{G}$

* L. c., p. 489.

† Comptes Rendus, vol. 156 (1913), p. 666.

are not tangent to curves of N or $\overline{N}$, the focal nets of $\overline{G}$ are a second pair of corresponding nets on S and $\overline{S}$. Consequently the asymptotic lines correspond on S and $\overline{S}$,* and $\overline{G}$ is a W congruence.

If N is to be a derived net of $\overline{N}$, we must have

$$x = \bar{x} + \bar{p}\frac{\partial \bar{x}}{\partial u} + \bar{q}\frac{\partial \bar{x}}{\partial v}, \tag{32}$$

and the point equation of N analogous to (31) is

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v}\log a p \bar{p}\frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u}\log b q \bar{q}\frac{\partial \theta}{\partial v}. \tag{33}$$

Comparing this equation with (1), we have $p\bar{p} = \overline{U}$, $q\bar{q} = \overline{V}$, where $\overline{U}$ and $\overline{V}$ are functions of u and v respectively. From (24) and (32) it is seen that the parameters u and v can be chosen so that

$$p\bar{p} = q\bar{q} = c', \tag{34}$$

where c' is a constant.

When the expression (24) for $\bar{x}$ is substituted in (32), the resulting equation is reducible to the form (2) where $r = -1$, and

$$\begin{aligned} -a' &= \frac{p}{c'} + \frac{1}{p} + \frac{1}{p}\frac{\partial p}{\partial u} + \left(\frac{q}{p} + \frac{p}{q}\right)\frac{\partial \log a}{\partial v} + \frac{1}{q}\frac{\partial p}{\partial v}, \\ -b' &= \frac{q}{c'} + \frac{1}{q} + \frac{1}{q}\frac{\partial q}{\partial v} + \left(\frac{p}{q} + \frac{q}{p}\right)\frac{\partial \log b}{\partial u} + \frac{1}{p}\frac{\partial q}{\partial u}. \end{aligned} \tag{35}$$

By means of (30) equations (35) are reducible to

$$\left(\frac{\partial^2\theta_1}{\partial v^2} + \frac{\partial^2\theta_1}{\partial u^2} - a'\frac{\partial\theta_1}{\partial u} + \frac{1}{c'}\theta_1\right)\frac{\partial\theta_2}{\partial v} = \left(\frac{\partial^2\theta_2}{\partial v^2} + \frac{\partial^2\theta_2}{\partial u^2} - a'\frac{\partial\theta_2}{\partial u} + \frac{1}{c'}\theta_2\right)\frac{\partial\theta_1}{\partial v},$$

$$\left(\frac{\partial^2\theta_1}{\partial v^2} + \frac{\partial^2\theta_1}{\partial u^2} - b'\frac{\partial\theta_1}{\partial v} + \frac{1}{c'}\theta_1\right)\frac{\partial\theta_2}{\partial u} = \left(\frac{\partial^2\theta_2}{\partial v^2} + \frac{\partial^2\theta_2}{\partial u^2} - b'\frac{\partial\theta_2}{\partial v} + \frac{1}{c'}\theta_2\right)\frac{\partial\theta_1}{\partial u},$$

from which it follows that θ_1 and θ_2 are solutions of

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = a'\frac{\partial\theta}{\partial u} + b'\frac{\partial\theta}{\partial v} + c'\theta. \tag{36}$$

In order that equations (1) and (2) with $r = -1$ may have three independent solutions, the functions P, Q and R in (9) must be equal to zero. Since θ_1 and θ_2 must be common solutions of (1) and (36), it follows then from (9) that $S = 0$. Hence from the last of (11) we have (17) and then from the first of (11) we get (18), that is N is a net R with equations reducible to (19). When these conditions are satisfied, so also are equa-

* E., p. 130.

tions (11) for (1) and (36), and consequently this system admits four independent solutions.

If we substitute in (24) the expression (32) for x, we find that in consequence of (34) and (35) the coördinates of $\bar{N}$ satisfy the equation

$$(37)\qquad \frac{\partial^2 \bar{\theta}}{\partial u^2} + \frac{\partial^2 \bar{\theta}}{\partial v^2} = 2\frac{\partial}{\partial u}\log ap\,\frac{\partial \bar{\theta}}{\partial u} + 2\frac{\partial}{\partial v}\log bq\,\frac{\partial \bar{\theta}}{\partial v}.$$

From (31) and (37) it follows that $\bar{N}$ is an R net.

We have just seen that if θ_1 and θ_2 are independent solutions of (1) and (36) the tangent planes to $\bar{N}$ pass through the corresponding points of N. In order that N be a derived net of $\bar{N}$ there must be two solutions of (31) such that $\bar{p}$ and $\bar{q}$ are of the forms (25). The latter are equivalent to the condition that θ_1 and θ_2 satisfy the relation

$$\theta + p\frac{\partial \theta}{\partial u} + q\frac{\partial \theta}{\partial v} = 0.$$

Hence in consequence of (34) there must exist two solutions of (31) such that

$$(38)\qquad \bar{\theta} + \frac{c'}{p}\frac{\partial \bar{\theta}}{\partial u} + \frac{c'}{q}\frac{\partial \bar{\theta}}{\partial v} = 0.$$

Differentiating this equation with respect to u and v and making use of (31) and (35) we obtain

$$\frac{\partial^2 \bar{\theta}}{\partial u^2} = \left(2\frac{\partial}{\partial u}\log ap + \frac{1}{p} + \frac{q}{p}\frac{\partial \log a}{\partial v}\right)\frac{\partial \bar{\theta}}{\partial u} - \frac{p}{q}\frac{\partial \log \bar{b}}{\partial u}\frac{\partial \bar{\theta}}{\partial v},$$

$$\frac{\partial^2 \bar{\theta}}{\partial v^2} = -\frac{q}{p}\frac{\partial \log a}{\partial v}\frac{\partial \bar{\theta}}{\partial u} + \left(2\frac{\partial}{\partial v}\log bq + \frac{1}{q} + \frac{p}{q}\frac{\partial \log b}{\partial u}\right)\frac{\partial \bar{\theta}}{\partial v}.$$

By differentiation we find that these equations are consistent with (31), and consequently there exist two independent solutions of (31) and (38). Hence we have the theorem:

If N is a net R whose coördinates satisfy (19), *each pair of solutions of* (1) *and*

$$(39)\qquad \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = 2\frac{\partial \log a}{\partial u}\frac{\partial \theta}{\partial u} + 2\frac{\partial \log b}{\partial v}\frac{\partial \theta}{\partial v} + c\theta,$$

where c is an arbitrary constant, determines an R net $\bar{N}$ without further quadratures such that N and $\bar{N}$ are derived nets of one another; and N and $\bar{N}$ are the focal nets of a W congruence.

Since equations (1) and (39) form a completely integrable system, any solution is expressible linearly in terms of four of them. Hence from the form of (25) we have the theorem:

An R net admits $\infty^5 W$ *transforms into R nets for each value of c in* (39).

6. **Theorem of permutability of transformations W.** Let θ_3 and θ_4 be two solutions of the equations

$$
(40)\qquad
\begin{aligned}
&\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial}{\partial v}\log a\cdot\frac{\partial\theta}{\partial u} + \frac{\partial}{\partial u}\log b\cdot\frac{\partial\theta}{\partial v},\\
&\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = 2\frac{\partial\log a}{\partial u}\frac{\partial\theta}{\partial u} + 2\frac{\partial\log b}{\partial v}\frac{\partial\theta}{\partial v} + c'\theta,
\end{aligned}
$$

where c' is a constant, and consider the functions

$$
(41)\qquad \bar{\theta}_i = \theta_i + p\frac{\partial\theta_i}{\partial u} + q\frac{\partial\theta_i}{\partial v}\qquad (i = 3, 4),
$$

where p and q are given by (25). Analogously to (26) we have

$$
(42)\qquad
\begin{aligned}
\frac{\partial\bar{\theta}_i}{\partial u} &= p\left\{\frac{\partial^2\theta_i}{\partial u^2} + \frac{1}{\Delta}\left[\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right.\right.\\
&\qquad\left.\left. - \frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\right]\right\}\\
\frac{\partial\bar{\theta}_i}{\partial v} &= q\left\{\frac{\partial^2\theta_i}{\partial v^2} + \frac{1}{\Delta}\left[\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial v^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial v^2}\right)\right.\right.\\
&\qquad\left.\left. - \frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial v^2} - \frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial v^2}\right)\right]\right\}\\
&= q\left\{-\frac{\partial^2\theta_i}{\partial u^2} + (c'-c)\theta_i + c\bar{\theta}_i - \frac{1}{\Delta}\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right.\\
&\qquad\left. + \frac{1}{\Delta}\frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\right\}\\
&= q\left\{-\frac{1}{p}\frac{\partial\bar{\theta}_i}{\partial u} + (c'-c)\theta_i + c\bar{\theta}_i\right\}.
\end{aligned}
$$

Differentiating the first of these expressions with respect to u, we get

$$
\begin{aligned}
\frac{\partial^2\bar{\theta}_i}{\partial u^2} = \frac{\partial\bar{\theta}_i}{\partial u}\left[\frac{\partial}{\partial u}\log pa^2b + \frac{1}{\Delta}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right] + p(c'-c)\frac{\partial\theta_i}{\partial u}\\
- p\frac{\partial\log b}{\partial u}\left[c'\theta_i + c\left(p\frac{\partial\theta_i}{\partial u} + q\frac{\partial\theta_i}{\partial v}\right)\right].
\end{aligned}
$$

Making use of the expression (30) for $\partial p/\partial u$, we reduce this equation to

$$
\begin{aligned}
\frac{\partial^2\bar{\theta}_i}{\partial u^2} = \frac{\partial\bar{\theta}_i}{\partial u}\left[\frac{\partial}{\partial u}\log p^2a^2b + \frac{1}{p} + \frac{q}{p}\frac{\partial\log a}{\partial v}\right] + p(c'-c)\frac{\partial\theta_i}{\partial u}\\
- p\frac{\partial\log b}{\partial u}[(c'-c)\theta_i + c\bar{\theta}_i].
\end{aligned}
$$

Also we find that

$$\frac{\partial^2\bar{\theta}_i}{\partial u\partial v} = \frac{\partial}{\partial v}\log ap \cdot \frac{\partial\bar{\theta}_i}{\partial u} + \frac{\partial}{\partial u}\log bq \cdot \frac{\partial\bar{\theta}_i}{\partial v}.$$

From the second of (42) we get

$$\frac{\partial^2\bar{\theta}_i}{\partial v^2} = -\frac{q}{p}\frac{\partial \log a}{\partial v}\frac{\partial\bar{\theta}_i}{\partial u} + \left(\frac{\partial \log q}{\partial v} - \frac{q}{p}\frac{\partial}{\partial u}\log bq + cq\right)\frac{\partial\bar{\theta}_i}{\partial v} + q(c' - c)\frac{\partial\theta}{\partial v}.$$

With the aid of the expressions (30) we obtain

$$\frac{\partial^2\bar{\theta}_i}{\partial u^2} + \frac{\partial^2\bar{\theta}_i}{\partial v^2} = 2\frac{\partial}{\partial u}\log ap\frac{\partial\bar{\theta}_i}{\partial u} + 2\frac{\partial}{\partial v}\log bq\frac{\partial\bar{\theta}_i}{\partial v} + c'\bar{\theta}_i.$$

Hence the functions $\bar{\theta}_3$ and $\bar{\theta}_4$ determine a W transform of $\bar{N}$. The coordinates of this transform $\hat{N}$ are of the form

$$\hat{x} = \bar{x} + \bar{p}\frac{\partial\bar{x}}{\partial u} + \bar{q}\frac{\partial\bar{x}}{\partial v}, \tag{43}$$

where $\partial\bar{x}/\partial u$ and $\partial\bar{x}/\partial v$ are given by (26), and

$$\begin{gathered}\bar{p} = \frac{1}{\bar{\Delta}}\left(\bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial v} - \bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial v}\right), \qquad \bar{q} = \frac{1}{\bar{\Delta}}\left(\bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial u} - \bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial u}\right), \\ \bar{\Delta} = \frac{\partial\bar{\theta}_4}{\partial u}\frac{\partial\bar{\theta}_3}{\partial v} - \frac{\partial\bar{\theta}_3}{\partial u}\frac{\partial\bar{\theta}_4}{\partial v}.\end{gathered} \tag{44}$$

From the above expressions we find

$$\begin{aligned}\bar{\Delta} &= (c' - c)\left(\theta_3\frac{\partial\bar{\theta}_4}{\partial u} - \theta_4\frac{\partial\bar{\theta}_3}{\partial u}\right) + c\left(\bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial u} - \bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial u}\right) \\ &= \frac{p}{\Delta}\left\{c'\left[\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_2}{\partial v}\frac{\partial\theta_1}{\partial u}\right)\left(\theta_3\frac{\partial^2\theta_4}{\partial u^2} - \theta_4\frac{\partial^2\theta_3}{\partial u^2}\right)\right.\right. \\ &\qquad + \left(\theta_3\frac{\partial\theta_4}{\partial u} - \theta_4\frac{\partial\theta_3}{\partial u}\right)\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right) \\ &\qquad \left. - \left(\theta_3\frac{\partial\theta_4}{\partial v} - \theta_4\frac{\partial\theta_3}{\partial v}\right)\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\right] \\ &\qquad + c\left[\left(\frac{\partial\theta_3}{\partial u}\frac{\partial\theta_4}{\partial v} - \frac{\partial\theta_4}{\partial u}\frac{\partial\theta_3}{\partial v}\right)\left(\theta_2\frac{\partial^2\theta_1}{\partial u^2} - \theta_1\frac{\partial^2\theta_2}{\partial u^2}\right)\right. \\ &\qquad + \left(\theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right)\left(\frac{\partial\theta_3}{\partial v}\frac{\partial^2\theta_4}{\partial u^2} - \frac{\partial\theta_4}{\partial v}\frac{\partial^2\theta_3}{\partial u^2}\right) \\ &\qquad \left.\left. - \left(\theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right)\left(\frac{\partial\theta_3}{\partial u}\frac{\partial^2\theta_4}{\partial u^2} - \frac{\partial\theta_4}{\partial u}\frac{\partial^2\theta_3}{\partial u^2}\right)\right]\right\}.\end{aligned} \tag{45}$$

By making use of first of these expressions for $\bar{\Delta}$, (24) and (26), we reduce (43) to the form

$$
\begin{aligned}
\hat{x} = x + \frac{c'-c}{\Delta\bar{\Delta}}\, p \Bigg\{ \frac{\partial x}{\partial u}\Bigg[& \left(\theta_1 \frac{\partial \theta_2}{\partial v} - \theta_2 \frac{\partial \theta_1}{\partial v}\right)\left(\theta_3 \frac{\partial^2 \theta_4}{\partial u^2} - \theta_4 \frac{\partial^2 \theta_3}{\partial u^2}\right) \\
& - \left(\theta_4 \frac{\partial \theta_3}{\partial v} - \theta_3 \frac{\partial \theta_4}{\partial v}\right)\left(\theta_2 \frac{\partial^2 \theta_1}{\partial u^2} - \theta_1 \frac{\partial^2 \theta_2}{\partial u^2}\right)\Bigg] \\
& - \frac{\partial x}{\partial v}\Bigg[\left(\theta_1 \frac{\partial \theta_2}{\partial u} - \theta_2 \frac{\partial \theta_1}{\partial u}\right)\left(\theta_3 \frac{\partial^2 \theta_4}{\partial u^2} - \theta_4 \frac{\partial^2 \theta_3}{\partial u^2}\right) \\
& - \left(\theta_4 \frac{\partial \theta_3}{\partial u} - \theta_3 \frac{\partial \theta_4}{\partial u}\right)\left(\theta_2 \frac{\partial^2 \theta_1}{\partial u^2} - \theta_1 \frac{\partial^2 \theta_2}{\partial u^2}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{46}
$$

In consequence of (45) the expression (46) is symmetrical in c and c', and in the pairs of functions θ_1, θ_2 and θ_3, θ_4. Hence the net $\hat{N}$ can be obtained also by applying to N the W transformation determined by θ_3 and θ_4, and then to the resulting net $\bar{N}$ the transformation determined by the functions

$$
\bar{\theta}_i = \theta_i + \frac{\left(\theta_4 \dfrac{\partial \theta_3}{\partial v} - \theta_3 \dfrac{\partial \theta_4}{\partial v}\right)\dfrac{\partial \theta_i}{\partial u} - \left(\theta_4 \dfrac{\partial \theta_3}{\partial u} - \theta_3 \dfrac{\partial \theta_4}{\partial u}\right)\dfrac{\partial \theta_i}{\partial v}}{\dfrac{\partial \theta_3}{\partial u}\dfrac{\partial \theta_4}{\partial v} - \dfrac{\partial \theta_4}{\partial u}\dfrac{\partial \theta_3}{\partial v}} \qquad (i = 1, 2),
$$

which are analogous to (41).

When $c' \neq c$, we have that $\hat{N}$ is a W transform of $\bar{N}$ and $\bar{N}$. Hence:

If N is an R net, and N_1 and N_2 are obtained from N by transformations W_{c_1} and W_{c_2}, there can be found directly an R net N_{12}, which is in relations of transformations W_{c_2}' and W_{c_1}' with N_1 and N_2 respectively.

When $c' = c$, $\hat{N}$ coincides with N. Hence:

If an R net $\bar{N}$ is a W transform of an R net N by means of solutions θ_1 and θ_2 of equations (39), and θ_3 and θ_4 are two other solutions of (39) independent of θ_1 and θ_2, then N is the W transform of $\bar{N}$ by means of the functions (41).

This result is a proof of the theorem at the end of §5.

7. **Transformations** T. If N is any net whatsoever with its point equation of the form (1), and N' is a parallel net, defined by (20); if also θ and θ' are corresponding solutions of (1) and (22), that is in the relation (27), then equations of the form

$$
x_1 = x - \frac{\theta}{\theta'} x' \tag{47}
$$

give the coördinates of a net N_1 whose point equation is

$$
\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v} \log \frac{a\tau}{\theta'} \frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u} \log \frac{b\sigma}{\theta'} \frac{\partial \theta}{\partial v}, \tag{48}
$$

where

$$\tau = h\theta - \theta', \qquad \sigma = l\theta - \theta'. \tag{49}$$

The developables of the congruence G of lines joining corresponding points of N and N_1 meet the surfaces on which these nets lie in the curves of the nets.*

If N' and N'' are two nets parallel to N, defined by equations of the form

$$\frac{\partial x'}{\partial u} = h_1 \frac{\partial x}{\partial u}, \qquad \frac{\partial x'}{\partial v} = l_1 \frac{\partial x}{\partial v}; \qquad \frac{\partial x''}{\partial u} = h_2 \frac{\partial x}{\partial u}, \qquad \frac{\partial x''}{\partial v} = l_2 \frac{\partial x}{\partial v},$$

where h_1, l_1 and h_2, l_2 are pairs of solutions of (21), if θ_1 and θ_2 are solutions of (1) and θ_1' and θ_2'' are the corresponding solutions of the point equations of N' and N'', that is

$$\frac{\partial \theta_1'}{\partial u} = h_1 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = l_1 \frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta_2''}{\partial u} = h_2 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2''}{\partial v} = l_2 \frac{\partial \theta_2}{\partial v},$$

the equations of the form

$$x_1 = x - \frac{\theta_1}{\theta_1'} x', \qquad x_2 = x - \frac{\theta_2}{\theta_2''} x'' \tag{50}$$

define two T transforms, N_1 and N_2, of N. It is readily seen that the functions

$$\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'} \theta_2', \qquad \theta_{21} = \theta_1 - \frac{\theta_2}{\theta_2''} \theta_1'', \tag{51}$$

where

$$\frac{\partial \theta_2'}{\partial u} = h_1 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'}{\partial v} = l_1 \frac{\partial \theta_2}{\partial v}; \qquad \frac{\partial \theta_1''}{\partial u} = h_2 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1''}{\partial v} = l_2 \frac{\partial \theta_1}{\partial v}, \tag{52}$$

are solutions of the point equations of N_1 and N_2 respectively.

Furthermore, equations of the forms

$$x_1''' = x'' - \frac{\theta_1''}{\theta_1'} x', \qquad x_2'''' = x' - \frac{\theta_2'}{\theta_2''} x'' \tag{53}$$

define nets N_1''' and N_2'''' parallel to N_1 and N_2 respectively. In fact,

$$\frac{\partial x_1'''}{\partial u} = h_{12} \frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_1'''}{\partial v} = l_{12} \frac{\partial x_1}{\partial v}; \qquad \frac{\partial x_2''''}{\partial u} = h_{21} \frac{\partial x_2}{\partial u}, \qquad \frac{\partial x_2''''}{\partial v} = l_{21} \frac{\partial x_2}{\partial v},$$

* The results of this section are established in the memoir M. We say that G is the conjugate congruence of the transformation T defined by (47).

where

$$
(54)\quad
\begin{aligned}
h_{12} &= \frac{h_1\theta_1'' - h_2\theta_1'}{\tau_1}, & l_{12} &= \frac{l_1\theta_1'' - l_2\theta_1'}{\sigma_1}, \\
& & \tau_1 &= h_1\theta_1 - \theta_1', \qquad \sigma_1 = l_1\theta_1 - \theta_1', \\
h_{21} &= \frac{h_2\theta_2' - h_1\theta_2''}{\tau_2}, & l_{21} &= \frac{l_2\theta_2' - l_1\theta_2''}{\sigma_2}, \\
& & \tau_2 &= h_2\theta_2 - \theta_2'', \qquad \sigma_2 = l_2\theta_2 - \theta_2''.
\end{aligned}
$$

Also if we write

$$
(55)\qquad \theta_{12}''' = \theta_2'' - \frac{\theta_1''}{\theta_1'}\theta_2', \qquad \theta_{21}'''' = \theta_1' - \frac{\theta_2'}{\theta_2''}\theta_1'',
$$

we have

$$
\frac{\partial\theta_{12}'''}{\partial u} = h_{12}\frac{\partial\theta_{12}}{\partial u}, \quad \frac{\partial\theta_{12}'''}{\partial v} = l_{12}\frac{\partial\theta_{12}}{\partial v}; \quad \frac{\partial\theta_{21}''''}{\partial u} = h_{21}\frac{\partial\theta_{21}}{\partial u}, \quad \frac{\partial\theta_{21}''''}{\partial v} = l_{21}\frac{\partial\theta_{21}}{\partial v}.
$$

It is readily shown that the following expressions for x_{12} are equal:

$$
x_{12} = x_1 - \frac{\theta_{12}}{\theta_{12}'''}x_1''' = x_2 - \frac{\theta_{21}}{\theta_{21}''''}x_2''''.
$$

These equations are of the form (50), and consequently x_{12}, y_{12}, z_{12} are the coördinates of a net N_{12} which is a T transfrom of N_1 and also of N_2. Since θ_2' and θ_1'' are determined only to within additive constants by (52), there are ∞^2 nets N_{12} which are T transforms of both N_1 and N_2.

8. **Transformations T of R nets.** Let N be an R net whose coördinates satisfy equations (19). A parallel net N' is defined by equations of the form

$$
(56)\qquad x' = \lambda\frac{\partial x}{\partial u} + \mu\frac{\partial x}{\partial v} + \nu\frac{\partial^2 x}{\partial u^2},
$$

where λ, μ and ν are determined by the condition that (20) hold. Making use of (5) we find that λ, μ and ν must satisfy the system of equations:

$$
(57)\quad
\begin{aligned}
\frac{\partial\lambda}{\partial u} &= h - \mu\frac{\partial\log a}{\partial v} + \nu\left(\frac{1}{a}\frac{\partial^2 a}{\partial v^2} + 2\frac{\partial\log a}{\partial u}\frac{\partial\log b}{\partial u} - 2\frac{\partial^2\log a}{\partial u^2}\right. \\
&\qquad\qquad \left. - 2\frac{\partial\log b}{\partial v}\frac{\partial\log a}{\partial v}\right), \\
\frac{\partial\lambda}{\partial v} &= -\lambda\frac{\partial\log a}{\partial v} - 2\mu\frac{\partial\log a}{\partial u} - \nu\left(\frac{\partial^2\log a}{\partial u\partial v} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u}\right), \\
\frac{\partial\mu}{\partial u} &= -\mu\frac{\partial\log b}{\partial u} - \nu\left(\frac{\partial^2\log b}{\partial u\partial v} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u}\right), \\
\frac{\partial\mu}{\partial v} &= l - \lambda\frac{\partial\log b}{\partial u} - 2\mu\frac{\partial\log b}{\partial v} - \frac{\nu}{b}\frac{\partial^2 b}{\partial u^2}, \\
\frac{\partial\nu}{\partial u} &= -\lambda - \nu\frac{\partial}{\partial u}\log ba^2, \qquad \frac{\partial\nu}{\partial v} = \mu - \nu\frac{\partial\log a}{\partial v}.
\end{aligned}
$$

From (47) we have by differentiation

$$(58)\qquad \frac{\partial x_1}{\partial u} = \frac{\tau}{\theta'^2}\left(x'\frac{\partial \theta}{\partial u} - \theta'\frac{\partial x}{\partial u}\right), \qquad \frac{\partial x_1}{\partial v} = \frac{\sigma}{\theta'^2}\left(x'\frac{\partial \theta}{\partial v} - \theta'\frac{\partial x}{\partial v}\right),$$

and

$$\frac{\partial^2 x_1}{\partial u^2} = \frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2}\frac{\partial x_1}{\partial u} + \frac{\tau}{\theta'^2}\left(x'\frac{\partial^2\theta}{\partial u^2} - \theta'\frac{\partial^2 x}{\partial u^2}\right),$$

$$\frac{\partial^2 x_1}{\partial v^2} = \frac{\partial}{\partial v}\log\frac{\sigma}{\theta'^2}\frac{\partial x_1}{\partial v} + \frac{\sigma}{\theta'^2}\left(x'\frac{\partial^2\theta}{\partial v^2} - \theta'\frac{\partial^2 x}{\partial v^2}\right).$$

By means of (56) and (58) these equations are reducible to

$$(59)\qquad \begin{aligned} \frac{\partial^2 x_1}{\partial u^2} &= \left(\frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2} - \frac{\lambda}{\nu}\right)\frac{\partial x_1}{\partial u} - \frac{\mu}{\nu}\frac{\tau}{\sigma}\frac{\partial x_1}{\partial v} + \frac{\tau x'}{\theta'^2}\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\lambda}{\nu}\frac{\partial\theta}{\partial u}\right. \\ &\qquad \left. + \frac{\mu}{\nu}\frac{\partial\theta}{\partial v} - \frac{\theta'}{\nu}\right), \\ \frac{\partial^2 x_1}{\partial v^2} &= \frac{\sigma}{\tau}\left(2\frac{\partial \log a}{\partial u} + \frac{\lambda}{\nu}\right)\frac{\partial x_1}{\partial u} + \left(\frac{\partial}{\partial v}\log\frac{\sigma b^2}{\theta'^2} + \frac{\mu}{\nu}\right)\frac{\partial x_1}{\partial v} \\ &\qquad + \frac{\sigma x'}{\theta'^2}\left[\frac{\partial^2\theta}{\partial v^2} - \left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right)\frac{\partial\theta}{\partial u} - \left(\frac{\mu}{\nu} + 2\frac{\partial \log b}{\partial v}\right)\frac{\partial\theta}{\partial v} + \frac{\theta'}{\nu}\right]. \end{aligned}$$

From (19) and (48) it follows that N_1 will be an R net if

$$(60)\qquad \frac{\partial^2 x_1}{\partial u^2} + \frac{\partial^2 x_1}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau}{\theta'}\frac{\partial x_1}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma}{\theta'}\frac{\partial x_1}{\partial v}.$$

Consequently we must have

$$(61)\qquad \begin{aligned} 2\frac{\partial}{\partial u}\log\frac{a\tau}{\theta'} &= \frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2} - \frac{\lambda}{\nu} + \frac{\sigma}{\tau}\left(2\frac{\partial\log a}{\partial u} + \frac{\lambda}{\nu}\right), \\ 2\frac{\partial}{\partial v}\log\frac{b\sigma}{\theta'} &= \frac{\partial}{\partial v}\log\frac{\sigma b^2}{\theta'^2} + \frac{\mu}{\nu} - \frac{\tau}{\sigma}\frac{\mu}{\nu}, \end{aligned}$$

and

$$(62)\qquad \begin{aligned} &\frac{\partial^2\theta}{\partial u^2} + \frac{\lambda}{\nu}\frac{\partial\theta}{\partial u} + \frac{\mu}{\nu}\frac{\partial\theta}{\partial v} - \frac{\theta'}{\nu} = \frac{n\sigma}{\nu}, \\ &\frac{\partial^2\theta}{\partial v^2} - \left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right)\frac{\partial\theta}{\partial u} - \left(\frac{\mu}{\nu} + 2\frac{\partial\log b}{\partial v}\right)\frac{\partial\theta}{\partial v} + \frac{\theta'}{\nu} = -\frac{n\tau}{\nu}, \end{aligned}$$

where n is a function to be determined.

Equations (61) are reducible by means of (49) to

$$(63)\qquad \frac{\partial h}{\partial u} = (l-h)\left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right), \qquad \frac{\partial l}{\partial v} = (l-h)\frac{\mu}{\nu}.$$

The function $(l-h)/\nu$ is found by differentiation to be a constant. Hence we have

$$l - h = m\nu, \tag{64}$$

where m is a constant.

If the first and second of (62) are differentiated with respect to v and u respectively, we find that n is an arbitrary constant. Adding equations (62), we get (39) with

$$c = mn. \tag{65}$$

Equations (21), (57) and (63) form a completely integrable system. If λ, μ, ν, h and l are multiplied by the same constant, the equations are satisfied and N' is replaced by a net homothetic to it, but the transform N_1 is the same. Hence there are four essential constants, and m is determined by (64). When one of these nets N' is known, each solution of (1) and (39) for a given c and n given by (65) determine the function θ' by means of (62). This transformation is unaltered if θ and θ' are multiplied by the same constant. Consequently the function θ involves only three essential constants in addition to c. Hence:

An R net N admits ∞^4 parallel nets determining congruences G of transformations T of N into R nets N_1; for each congruence there are ∞^4 of these nets N_1.

9. **Theorem of permutability of transformations T of R nets.** Let N_1 and N_2 be R nets obtained from an R net N by means of transformations T for which the functions are λ_i, μ_i, ν_i, h_i, l_i, m_i and n_i $(i = 1, 2)$. We desire to find the nets N_{12}, as defined in § 7, which are R nets.

From (51) we have by differentiation

$$\frac{\partial \theta_{12}}{\partial u} = \frac{\tau_1}{\theta_1'^2}\left(\theta_2'\frac{\partial \theta_1}{\partial u} - \theta_1'\frac{\partial \theta_2}{\partial u}\right), \qquad \frac{\partial \theta_{12}}{\partial v} = \frac{\sigma_1}{\theta_1'^2}\left(\theta_2'\frac{\partial \theta_1}{\partial v} - \theta_1'\frac{\partial \theta_2}{\partial v}\right)$$

and

$$\frac{\partial^2 \theta_{12}}{\partial u^2} = \frac{\tau_1}{\theta_1'^2}\left[\theta_2'\frac{\partial^2 \theta_1}{\partial u^2} - \theta_1'\frac{\partial^2 \theta_2}{\partial u^2} + \frac{\partial}{\partial u}\log\frac{\tau_1}{\theta_1'^2}\left(\theta_2'\frac{\partial \theta_1}{\partial u} - \theta_1'\frac{\partial \theta_2}{\partial u}\right)\right],$$

$$\frac{\partial^2 \theta_{12}}{\partial v^2} = \frac{\sigma_1}{\theta_1'^2}\left[\theta_2'\frac{\partial^2 \theta_1}{\partial v^2} - \theta_1'\frac{\partial^2 \theta_2}{\partial v^2} + \frac{\partial}{\partial v}\log\frac{\sigma_1}{\theta_1'^2}\left(\theta_2'\frac{\partial \theta_1}{\partial v} - \theta_1'\frac{\partial \theta_2}{\partial v}\right)\right].$$

From (60) it follows that the equation analogous to (39) is

$$\frac{\partial^2 \theta_{12}}{\partial u^2} + \frac{\partial^2 \theta_{12}}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1'}\frac{\partial \theta_{12}}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma_1}{\theta_1'}\frac{\partial \theta_{12}}{\partial v} + c_1\theta_{12}. \tag{66}$$

If we substitute the above expressions and take

$$c_1 = n_2 m_2, \tag{67}$$

the resulting equation is reducible to

$$(68)\quad m_1\left[\left(\lambda_1 - \lambda_2\frac{\nu_1}{\nu_2}\right)\frac{\partial\theta_2}{\partial u} + \left(\mu_1 - \mu_2\frac{\nu_1}{\nu_2}\right)\frac{\partial\theta_2}{\partial v} - \theta_2''\frac{\nu_1}{\nu_2}(n_2 - 1)\right] + n_2\frac{\theta_2}{\nu_2}(l_1h_2 - l_2h_1) + (n_2m_2 - m_1)\theta_2' = 0.$$

When in like manner we require that

$$\frac{\partial^2\theta_{21}}{\partial u^2} + \frac{\partial^2\theta_{21}}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau_2}{\theta_2''}\frac{\partial\theta_{21}}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma_2}{\theta_2''}\frac{\partial\theta_{21}}{\partial v} + n_1m_1\theta_{21},$$

we obtain

$$(69)\quad m_2\left[\left(\lambda_2 - \lambda_1\frac{\nu_2}{\nu_1}\right)\frac{\partial\theta_1}{\partial u} + \left(\mu_2 - \mu_1\frac{\nu_2}{\nu_1}\right)\frac{\partial\theta_1}{\partial v} - \theta_1'\frac{\nu_2}{\nu_1}(n_1 - 1)\right] + n_1\frac{\theta_1}{\nu_1}(l_2h_1 - l_1h_2) + (n_1m_1 - m_2)\theta_1'' = 0.$$

For the net N_1''', parallel to N_1, as given by (53), we have

$$(70)\quad x_1''' = \lambda_{12}\frac{\partial x_1}{\partial u} + \mu_{12}\frac{\partial x_1}{\partial v} + \nu_{12}\frac{\partial^2 x_1}{\partial u^2},$$

and the functions h_{12} and l_{12} are given by (54). The equations analogous to (63) are

$$\frac{\partial h_{12}}{\partial u} = (l_{12} - h_{12})\left(\frac{\lambda_{12}}{\nu_{12}} + 2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1'}\right), \qquad \frac{\partial l_{12}}{\partial v} = (l_{12} - h_{12})\frac{\mu_{12}}{\nu_{12}}.$$

On substituting the expressions (54) in these equations we get

$$(71)\quad \begin{aligned} (l_{12} - h_{12})\left(\frac{\lambda_{12}}{\nu_{12}} + \frac{\partial}{\partial u}\log\frac{\tau_1}{\theta_1'^2}\right) &= \frac{\theta_1'}{\tau_1}\left(m_1\lambda_1\frac{l_2\theta_1 - \theta_1''}{\sigma_1} - m_2\lambda_2\right), \\ (l_{12} - h_{12})\frac{\mu_{12}}{\nu_{12}} &= \frac{\theta_1'}{\sigma_1}\left(m_1\mu_1\frac{l_2\theta_1 - \theta_1''}{\sigma_1} - m_2\mu_2\right). \end{aligned}$$

Equating the expressions (53) and (70) for x_1''', we get

$$\lambda_{12}\frac{\partial x_1}{\partial u} + \mu_{12}\frac{\partial x_1}{\partial v} + \nu_{12}\frac{\partial^2 x_1}{\partial u^2} = \left(\lambda_2 - \frac{\theta_1''}{\theta_1'}\lambda_1\right)\frac{\partial x}{\partial u} + \left(\mu_2 - \frac{\theta_1''}{\theta_1'}\mu_1\right)\frac{\partial x}{\partial v} + \left(\nu_2 - \frac{\theta_1''}{\theta_1'}\nu_1\right)\frac{\partial^2 x}{\partial u^2}.$$

This equation is consistent with (71) when (69) is satisfied.

The equations analogous to (62) are

$$\frac{\partial^2\theta_{12}}{\partial u^2}+\frac{\lambda_{12}}{\nu_{12}}\frac{\partial\theta_{12}}{\partial u}+\frac{\mu_{12}}{\nu_{12}}\frac{\partial\theta_{12}}{\partial v}-\frac{\theta_{12}'''}{\nu_{12}}=\frac{n_2}{\nu_{12}}(l_{12}\theta_{12}-\theta_{12}'''),$$

$$\frac{\partial^2\theta_{12}}{\partial v^2}-\left(\frac{\lambda_{12}}{\nu_{12}}+2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1}\right)\frac{\partial\theta_{12}}{\partial u}-\left(\frac{\mu_{12}}{\nu_{12}}+2\frac{\partial}{\partial v}\log\frac{b\sigma_1}{\theta_1}\right)\frac{\partial\theta_{12}}{\partial v}+\frac{\theta_{12}'''}{\nu_{12}}$$
$$=-\frac{n_2}{\nu_{12}}(h_{12}\theta_{12}-\theta_{12}''').$$

These equations are satisfied also when the above expressions are used and also (51) and (55). Hence the problem reduces to the determination of the conditions when (68) and (69) are satisfied. There are two cases to be considered, according as m_1n_1 and m_2n_2 are equal or not.

When $m_1n_1=m_2n_2$, we have from equations of the form (62)

$$\left(\frac{\lambda_1}{\nu_1}-\frac{\lambda_2}{\nu_2}\right)\frac{\partial\theta_2}{\partial u}+\left(\frac{\mu_1}{\nu_1}-\frac{\mu_2}{\nu_2}\right)\frac{\partial\theta_2}{\partial v}-\frac{\theta_2'}{\nu_1}+\frac{\theta_2''}{\nu_2}=\frac{n_1}{\nu_1}(l_1\theta_2-\theta_2')$$
$$-\frac{n_2}{\nu_2}(l_2\theta_2-\theta_2''),$$

$$\left(\frac{\lambda_2}{\nu_2}-\frac{\lambda_1}{\nu_1}\right)\frac{\partial\theta_1}{\partial u}+\left(\frac{\mu_2}{\nu_2}-\frac{\mu_1}{\nu_1}\right)\frac{\partial\theta_1}{\partial v}-\frac{\theta_1''}{\nu_2}+\frac{\theta_1'}{\nu_1}=\frac{n_2}{\nu_2}(l_2\theta_1-\theta_1'')$$
$$-\frac{n_1}{\nu_1}(l_1\theta_1-\theta_1').$$

In consequence of these relations equations (68) and (69) are satisfied identically, and consequently each of the ∞^2 nets N_{12} is an R net. The determination of these nets requires the finding of θ_2' and θ_1'' from (52) by quadratures.

When $m_1n_1\neq m_2n_2$, we find by differentiation that the left-hand members of (68) and (69) are constant. Hence when θ_2' and θ_1'' are given the expressions obtained from (68) and (69), the resulting N_{12} is an R net. Therefore we have the theorem:

If N is an R net, and N_1 and N_2 are two T transforms of N by means of functions θ_1 and θ_2 which are solutions of (1) and (39) for the same value of c, all of the ∞^2 nets N_{12} are R nets, and their determination requires two quadratures; when the constant c in (39) is different for θ_1 and θ_2, there is a unique net N_{12} which is an R net and it can be found without quadrature.

10. **Permutability of transformations W and T of R nets.** Let N be any net, $\bar{N}$ its derived net determined by solutions θ_1 and θ_2 of the point equation (1) of N, and N_3 the T transform of N defined by equations of the form

$$x_3=x-\frac{\theta_3}{\theta_3'''}x''',$$

where θ_3 is a solution of (1), and $N'''(x''')$ is a net parallel to N, so that

$$\frac{\partial x'''}{\partial u} = h_3 \frac{\partial x}{\partial u}, \qquad \frac{\partial x'''}{\partial v} = l_3 \frac{\partial x}{\partial v}; \qquad \frac{\partial \theta_3'''}{\partial u} = h_3 \frac{\partial \theta_3}{\partial u}, \qquad \frac{\partial \theta_3'''}{\partial v} = l_3 \frac{\partial \theta_3}{\partial v}.$$

If θ_1''' and θ_2''' are defined by

$$(72) \qquad \frac{\partial \theta_1'''}{\partial u} = h_3 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'''}{\partial v} = l_3 \frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta_2'''}{\partial u} = h_3 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'''}{\partial v} = l_3 \frac{\partial \theta_2}{\partial v},$$

the functions

$$(73) \qquad \theta_{31} = \theta_1 - \frac{\theta_3}{\theta_3'''} \theta_1''' \qquad \theta_{32} = \theta_2 - \frac{\theta_3}{\theta_3'''} \theta_2'''$$

are solutions of the point equation of N_3. They determine a derived net $\bar{N}_3$ of N_3 whose equations are of the form

$$(74) \qquad \bar{x}_3 = x_3 + p_3 \frac{\partial x_3}{\partial u} + q_3 \frac{\partial x_3}{\partial v},$$

where

$$p_3 = \frac{1}{\Delta_3}\left(\theta_{31} \frac{\partial \theta_{32}}{\partial v} - \theta_{32} \frac{\partial \theta_{31}}{\partial v}\right), \qquad q_3 = \frac{1}{\Delta_3}\left(\theta_{32} \frac{\partial \theta_{31}}{\partial u} - \theta_{31} \frac{\partial \theta_{32}}{\partial u}\right),$$

$$\Delta_3 = \frac{\partial \theta_{32}}{\partial u}\frac{\partial \theta_{31}}{\partial v} - \frac{\partial \theta_{32}}{\partial v}\frac{\partial \theta_{31}}{\partial u}.$$

From (73) we have

$$\left.\begin{aligned} \frac{\partial \theta_{3i}}{\partial u} &= \frac{\theta_3 h_3 - \theta_3'''}{\theta_3'''^2}\left(\theta_i''' \frac{\partial \theta_3}{\partial u} - \theta_3''' \frac{\partial \theta_i}{\partial u}\right), \\ \frac{\partial \theta_{3i}}{\partial v} &= \frac{\theta_3 l_3 - \theta_3'''}{\theta_3'''^2}\left(\theta_i''' \frac{\partial \theta_3}{\partial v} - \theta_3''' \frac{\partial \theta_i}{\partial v}\right). \end{aligned}\right\} \quad (i = 1, 2).$$

Also we have

$$\frac{\partial x_3}{\partial u} = \frac{\theta_3 h_3 - \theta_3'''}{\theta_3'''^2}\left(x''' \frac{\partial \theta_3}{\partial u} - \theta_3''' \frac{\partial x}{\partial u}\right),$$

$$\frac{\partial x_3}{\partial v} = \frac{\theta_3 l_3 - \theta_3'''}{\theta_3'''^2}\left(x''' \frac{\partial \theta_3}{\partial v} - \theta_3''' \frac{\partial x}{\partial v}\right).$$

On substituting these expressions in (74), the resulting equation is reducible to

$$
\begin{aligned}
\bar{x}_3 = x + \frac{1}{\Delta'}\Bigg\{ x'''\Bigg[\theta_1\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_3}{\partial v} - \frac{\partial\theta_3}{\partial u}\frac{\partial\theta_2}{\partial v}\right) \\
+ \theta_2\left(\frac{\partial\theta_3}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_1}{\partial u}\frac{\partial\theta_3}{\partial v}\right) + \theta_3\left(\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v} - \frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v}\right)\Bigg] \\
+ \frac{\partial x}{\partial u}\Bigg[(\theta_2\theta_1''' - \theta_1\theta_2''')\frac{\partial\theta_3}{\partial v} + (\theta_3\theta_2''' - \theta_2\theta_3''')\frac{\partial\theta_1}{\partial v} \\
+ (\theta_1\theta_3''' - \theta_3\theta_1''')\frac{\partial\theta_2}{\partial v}\Bigg] - \frac{\partial x}{\partial v}\Bigg[(\theta_2\theta_1''' - \theta_1\theta_2''')\frac{\partial\theta_3}{\partial u} \\
+ (\theta_3\theta_2'' - \theta_2\theta_3''')\frac{\partial\theta_1}{\partial u} + (\theta_1\theta_3''' - \theta_3\theta_1''')\frac{\partial\theta_2}{\partial u}\Bigg]\Bigg\},
\end{aligned}
\tag{75}
$$

where

$$
\Delta' = \theta_2'''\left(\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_3}{\partial v} - \frac{\partial\theta_1}{\partial v}\frac{\partial\theta_3}{\partial u}\right) + \theta_1'''\left(\frac{\partial\theta_2}{\partial v}\frac{\partial\theta_3}{\partial u} - \frac{\partial\theta_2}{\partial u}\frac{\partial\theta_3}{\partial v}\right) + \theta_3'''\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_2}{\partial v}\frac{\partial\theta_1}{\partial u}\right).
$$

In § 4 it is shown that the equations of the form

$$
\bar{x}''' = x''' + p'''\frac{\partial x'''}{\partial u} + q'''\frac{\partial x'''}{\partial v},
$$

where

$$
p''' = \theta_1'''\frac{\partial\theta_2'''}{\partial v} - \theta_2'''\frac{\partial\theta_1'''}{\partial v}, \qquad q''' = \theta_2'''\frac{\partial\theta_1'''}{\partial u} - \theta_1'''\frac{\partial\theta_2'''}{\partial u},
$$

$$
\Delta''' = \left(\frac{\partial\theta_2'''}{\partial u}\frac{\partial\theta_1'''}{\partial v} - \frac{\partial\theta_1'''}{\partial v}\frac{\partial\theta_2'''}{\partial u}\right),
$$

define a net $\bar{N}'''$ parallel to $\bar{N}$. The functions

$$
\bar{\theta}_3 = \theta_3 + p\frac{\partial\theta_3}{\partial u} + q\frac{\partial\theta_3}{\partial v}, \qquad \bar{\theta}_3''' = \theta_3''' + p'''\frac{\partial\theta_3'''}{\partial u} + q'''\frac{\partial\theta_3'''}{\partial v}
$$

are corresponding solutions of the point equations of $\bar{N}$ and $\bar{N}'''$. Hence expressions of the form

$$
\bar{x} - \frac{\bar{\theta}_3}{\bar{\theta}_3'''}\bar{x}'''
$$

determine a T transform of $\bar{N}$. When the above expressions for these functions are substituted, the result is reducible to the right-hand member of (75). Since θ_1''' and θ_2''', as defined by (72) involve additive arbitrary constants, there are ∞^2 of the nets $\bar{N}_3$. Hence we have the theorem:

If $\bar{N}$ is a derived net of any net N_1 and N_3 is any T transform of N,

there can be found by two quadratures ∞^2 nets $\overline{N}_3$, each of which is a derived net of N_3 and a T transform of $\overline{N}$.

We apply this theorem true for any net to the case when N is an R net. Let $\overline{N}$ be the R net obtained from N by two solutions θ_1 and θ_2 of equations (1) and (39), and let N_3 be a T transform of N in accordance with the results of § 8. From equations analogous to those of § 9 it follows that the functions (73) satisfy equations for N_3 analogous to (39). Moreover, if θ_3 is a solution of (39), then θ_1''' and θ_2''' are determined by quadratures and involve additive constants; but if θ_3 is a solution of the equation obtained by replacing c by c' in (39), then θ_1''' and θ_2''' are uniquely determined. Hence:

If $\overline{N}$ is a W transform of an R net N by means of solutions θ_1 and θ_2 of (1) and (39), and N_3 is an R net which is a T transform by means of a function θ_3, a solution of (1) and (39) with c replaced by c', there can be found directly a unique R net $\overline{N}_3$ which is a W transform of N_3 and a T transform of $\overline{N}$; when $c' = c$, there are ∞^2 such nets $\overline{N}_3$, obtained by two quadratures.

PRINCETON UNIVERSITY.

THE APPLICATION OF MODERN THEORIES OF INTEGRATION TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

By T. C. Fry.

1. **Introduction.** It is the purpose of this paper to present a method of applying the modern theories of divergent and Stieltjes' integrals to the discussion of certain integrals closely related to the Fourier identity; and to present an application of this method to the solution of linear differential equations. The investigation results in assigning a meaning to a wide class of integrals which have heretofore had none; and in justifying the use of the common operations in dealing with these integrals.

In an attempt to obtain the solution of a class of electrical problems the writer was led, largely through physical arguments, to adopt formal operational methods of manipulation for the purpose of obtaining tentative results. As the work proceeded he was impressed with the large number of cases in which these methods seemed to yield correct answers, although so far as he knew they had no mathematical justification. In an attempt to remove the uncertainty in his own mind as to the breadth of the class of problems in which these methods might be used with a reasonable certainty of accurate results, he was led to formulate the argument which is presented in the following pages.

This being the origin of the work, it will not be at all surprising if it has influenced the form in which the presentation is cast. In fact, the paper is to a certain extent a companion to a technical article on "The Solution of Circuit Problems," which recently appeared in the Physical Review. Bearing in mind the fact that readers of that article may wish to refer to this, an attempt has been made to use as simple lines of argument as are consistent with rigor. This attempt has seemed to require more restrictive conditions upon some of the preliminary theorems than would otherwise have been necessary; but has left them in all cases sufficiently broad for the purposes of this discussion.

Broadly speaking, there are three main divisions of the argument. In the first, which comprises sections 2 to 7, certain concepts which result from the application of the Cæsaro definition of a divergent limit to integrals are presented. In the second division, which comprises sections 8 to 10, some observations are made regarding Stieltjes' integrals which depend upon an arbitrary parameter. In the third broad division, comprising the remainder of the paper, these two discussions, which have been

carried on with very little reference to one another, are both merged in the discussion of a type of integral closely resembling the Fourier integral. This is assigned a meaning, and applied to the solution of linear differential equations.

It is perhaps desirable to add a word in acknowledgment of the sources of information which have influenced the development of the argument. Foremost among these have been* Borel's treatise on Divergent Series, a few paragraphs from Stieltjes' original work on Continued Fractions, and Hildebrandt's excellent review of the Modern Theories of Generalized Integrals. Other sources have been consulted at various times, but have exerted much less influence.

2. **The Cæsaro value for a divergent limit.** The functions to which attention will be directed in the following pages are frequently expressed in the form of limits which do not exist in the ordinary sense. It is necessary, therefore, to assign values to these limits by definition. This is best done by finding a transformation I, possessing the property that $\text{Lim}\, If(u)$ exists under circumstances which render $\text{Lim}\, f(u)$ meaningless, and also the property that $\text{Lim}\, If(u) \equiv \text{Lim}\, f(u)$ whenever the latter limit exists. If such a transformation may be found, the identity $\text{Lim}\, If(u) \equiv \text{Lim}\, f(u)$ may be used as a definition of its right-hand member.

It has been shown that† the operator

$$I = \frac{1}{n}\int_0^n dn \tag{1}$$

possesses the necessary properties, when n is to approach ∞. That is,

$$\underset{n \doteq \infty}{\text{Lim}} \frac{1}{n}\int_0^n f(n)dn = \underset{n \doteq \infty}{\text{Lim}} f(n), \tag{2}$$

whenever the limit of $f(n)$ exists; while the right-hand side of (2) fails to have a meaning for many functions for which the left-hand side exists. Furthermore, the statements

$$I(u + v) = I(u) + I(v), \qquad I(cu) = cI(u), \qquad I(c) = c \tag{3}$$

are almost immediately obvious.

For the purposes of this paper it is desirable to limit consideration to functions of the type,

$$f(n) = \int_0^n e^{inx}\phi(n)dn, \tag{4}$$

* Borel, "Lecons sur les Series Divergentes."

Stieltjes, "Sur les Fractions Continues," Annales de la Faculte' des Sciences de Toulouse, 1894–5.

Hildebrandt, "On Integrals," Bull. Amer. Math. Soc., 1917–18.

† Borel, loc. cit., p. 87.

Silverman, Transactions of Am. Math. Soc., Apr., 1916.

which are peculiar in that their Cæsaro limits can be evaluated by means of the theory of residues.

3. **Expressions of the type**

$$f(n) = \int_0^n n^j e^{inx} dn. \tag{5}$$

If j is integral the operator I may be applied to expressions such as (5). The transformed functions are then obtained by the use of the ordinary formulæ of integral calculus. Direct integration of (5) leads to an expression

$$f(n) = \sum_{k=0}^{j} a_k n^k e^{inx} - a_0, \tag{6}$$

and therefore to

$$If(n) = \sum_{k=0}^{j} a_k I n^k e^{inx} - a_0. \tag{7}$$

The terms of this series are all of the form $In^k e^{inx}$, which can be evaluated by immediate integration. The result is

$$In^k e^{inx} = \sum_{k'=1}^{k} a_{k'} n^{k'-1} e^{inx} + a_{0'} \frac{e^{inx} - 1}{n}, \tag{8}$$

where the constants depend not only on k' but also on k. It is the form of this expression, however, rather than the actual values of its coefficients, which is important.

If (8) is substituted in (7) a new expression results which may be thrown into the form

$$If(n) = \sum_{k=1}^{j} a_k' n^{k-1} e^{inx} + a_0' \frac{e^{inx} - 1}{n} - a_0.$$

Repeated application of I leads at last to

$$I^{j+1} f(n) = \sum_{k=1}^{j+1} a_0^{(k)} I^{j-k+1} \frac{e^{inx} - 1}{n} - a_0,$$

whence, by passing to the limit as $n = \infty$,

$$\operatorname*{Lim}_{n=\infty} I^{j+1} f(n) = - a_0,$$

since

$$\operatorname*{Lim}_{n=\infty} I^{j-k+1} \frac{e^{inx} - 1}{n} = \operatorname*{Lim}_{n=\infty} \frac{e^{inx} - 1}{n} = 0.$$

The meaning of the integral $\int_0^\infty n^j e^{inx} dn$ is therefore known as soon as the value of a_0 has been determined. This leads at once to the result

$$\int_0^\infty n^j e^{inx} dn = (-1)^{j+1} \frac{\underline{|j}}{(ix)^{j+1}}. \tag{9}$$

This integral was evaluated along the real axis. It will be found illuminating to obtain its value when taken along certain other paths of integration. Suppose, for instance, it is evaluated along a line which makes an angle θ with the real axis, as shown in Fig. 1. To be explicit,

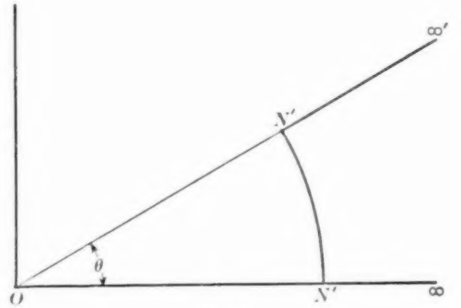

FIG. 1.

it is assumed that $|\theta| \leqq \pi/2$ and that* $\operatorname{sgn} \theta = \operatorname{sgn} x$. Along this path the integral is convergent since $\operatorname*{Lim}_{n \doteq \infty} e^{inx} = 0$. Evaluating (6) it is found that

$$\operatorname*{Lim}_{n \doteq \infty'} f(n) = -a_0,$$

which is the same as (9). This equality of the values of the integral (5) when taken along the two paths $O\infty$ and $O\infty'$ will be used in several different forms in the succeeding sections. In the notation used in Fig. 1 they may be symbolically expressed as

$$\begin{gathered}\int_{\infty' 0 \infty} = 0, \qquad \int_{\infty' N' N \infty} = 0, \qquad \int_{0 N N' \infty'} = \int_{0 \infty}, \\ \int_{N N'} = -\int_{N' 0 N}, \qquad \operatorname*{Lim}_{N = |N| \doteq \infty} \int_{N N'} = 0.\end{gathered} \tag{10}$$

This argument has been carried out on the assumption that j is a positive integer. There is, however, no essential difficulty introduced when j is not integral, provided it is greater than -1. The course of the argument is unaltered except that $-a_0$ is replaced by an ordinary convergent improper integral. The value is easily determined to be $(-1)^{j+1}\Gamma(j+1)/(ix)^{j+1}$. Formula (10) also applies to this more general case.

4. **Expressions of the Type (4).** The argument concerning the general equation (4) can be made to depend upon the results obtained in section 3, provided the function $\phi(n)$ has no essential singularity at infinity. There

* The figure is drawn for $\operatorname{sgn} x = 1$.

may be essential singularities in the finite portion of the plane, but if so, it will be assumed that they are so situated that it is possible to connect them by cuts which do not anywhere intersect the real axis.* If $\phi(n)$ has at infinity a pole of order ν, it may be expanded in the series $\phi(n) = \sum_{j=-\nu}^{\infty} a_j/n^j$, which converges for all values of n satisfying the condition $|n| \geqq N$. Furthermore, the series formed from the terms $(a_j/n^j)e^{inx}$ also converges in the same region uniformly and absolutely if the imaginary part of nx is not negative. This establishes the propriety of term by term integration; hence

$$\int_N^n \phi(n)e^{inx}dn = \sum_{j=-\nu}^{\infty} a_j \int_N^n \frac{e^{inx}}{n^j}\,dn.$$

Now introduce the notation

$$\phi(N, n) = \sum_{j=1}^{\infty} a_j \int_N^n \frac{e^{inx}}{n^j}\,dn.$$

Then $\operatorname*{Lim}_{n \doteq \infty} \Phi(N, n)$ converges to the value $\Phi(N, \infty)$ in the ordinary sense of convergence. In terms of this notation (5) may be expressed as

$$\begin{aligned} \int_N^\infty \phi(n)e^{inx}dn &= \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1} \int_N^n \phi(n)e^{inx}dn \\ &= \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1}\Phi(N, n) + \sum_{j=-\nu}^{0} a_j \operatorname*{Lim}_{n \doteq \infty} I^{\nu+1} \int_N^n \frac{e^{inx}}{n^j}\,dn \qquad (11) \\ &= \Phi(N, \infty) - \sum_{j=-\nu}^{0} a_j \left[\int_0^N \frac{e^{inx}}{n^j}\,dn + \frac{|\underline{-j}}{(ix)^{-j}} \right]. \end{aligned}$$

Since this equation gives the value of the limit (11) it also asserts its existence. It is possible, however, to simplify its calculation by establishing its equivalence with a certain Cauchy integral. This will be the object of the next section.

5. **Reduction to a closed path: Cauchy integral.** The integration in equation (11) was performed along the real axis. If it had been carried out along the line $ON'\infty'$ of Fig. 1 there would have been no need of applying the operator I. The result of the integration along this line could have been determined immediately to be

$$\int_{N'}^{\infty'} \phi(n)e^{inx}dn = \Phi(N', \infty') - \sum_{j=-\nu}^{0} a_j \left[\int_0^{N'} \frac{e^{inx}}{n^j}\,dn + \frac{|\underline{-j}}{(ix)^{-j}} \right].$$

* We are aiming at an evaluation of

$$\int \phi(n)e^{inx}dn \qquad (a)$$

taken along the real axis and the condition is stated for this particular case. If we desired to carry out the integration of (a) along some path C it is only necessary to require that no cuts joining the essential singularities of $\phi(n)$ shall intersect C at any point, finite or infinite.

Hence

$$\int_{\infty' N' N \infty} \phi(n)e^{inx}dn = \Phi(N, \infty) - \Phi(N', \infty') - \sum_{j=1}^{\infty} a_j \int_N^{N'} \frac{e^{inx}}{n^j} dn. \quad (12)$$

The value of the left-hand side of (12) is independent of the magnitude of $N = |N'|$ so long as N is sufficiently large. The same is therefore true of the right-hand side also. Call this value L. Then as N is indefinitely increased the limit approached by both members of (12) must be this value L to which they are constantly equal. But it is almost immediately obvious that the limits of $\Phi(N, \infty)$ and $\Phi(N', \infty')$ are zero since $\lim_{n\,\infty} \Phi(N, n)$ converges in the ordinary sense. As for the summation, it is seen that excluding the term $j = 1$,

$$\left|\sum_{j=2}^{\infty} a_j \int_N^{N'} \frac{e^{inx}}{n^j} dn\right| \leqq \sum_{j=2}^{\infty} \frac{|a_j|}{N^{j-1}} |\theta|,$$

which also approaches the limit zero. This leaves for consideration only the term $j = 1$. However, since $|\theta| \leqq \pi/2$,

$$\left|a_1 \int_N^{N'} e^{inx} \frac{dn}{n}\right| \leqq |a_1| \left|\int_0^{\theta} e^{-xN\theta/2} d\theta\right| \leqq \frac{2|a_1|}{Nx},$$

which also approaches the limit zero. That is, $L = 0$ and hence

$$\int_{\infty' N' N \infty} \phi(n)e^{inx}dn = 0, \qquad \int_{0 N N' \infty'} \phi(n)e^{inx}dn = \int_{0\infty} \phi(n)e^{inx}dn. \quad (13)$$

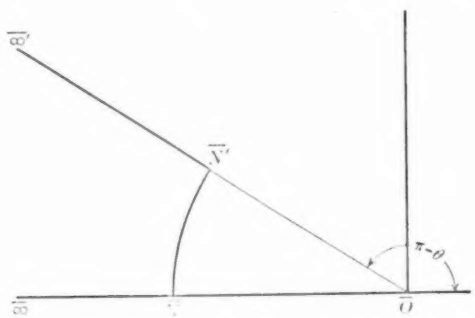

FIG. 2.

It is quite obvious that a similar argument may be applied to the integral

$$\int_{-\infty}^{0} \phi(n)e^{inx}dn.$$

In the notation of Fig. 2 the results of such an argument are

$$\int_{\overline{\infty}' \overline{N}' \overline{N} \overline{\infty}} \phi(n)e^{inx}dn = 0, \qquad \int_{\overline{\infty}' \overline{N}' \overline{N} \overline{0}} \phi(n)e^{inx}dn = \int_{\overline{\infty}\overline{0}} \phi(n)e^{inx}dn. \quad (14)$$

Now let $\theta = \pi/2$ in both (13) and (14); then $\bar{\infty}' = \infty'$ and $\bar{N}' = N'$ so that in Figs. (1) and (2) the paths $\infty' N'$ and $\bar{\infty}'\bar{N}'$ either coincide or lie vertically above one another on different sheets of the Riemann surface for $\phi(n)$. The conditions which have been imposed upon the singularities of ϕ are sufficient, however, to justify the statement that if the points O and $\bar{O}$ lie upon the same sheet of this Riemann surface the points $\bar{N}'$ and N' will also lie upon the same sheet, so that the integrals along the paths $\bar{\infty}'\bar{N}'$ and $\infty' N'$ are equal. This results finally in the equation

$$\int_{-\infty}^{\infty} \phi(n) e^{inx} dn = \int_{N'\bar{N}\bar{O}NN'} \phi(n) e^{inx} dn. \tag{15}$$

That is, in words:

THEOREM 1. *The Caesaro value of the integral*

$$\int_{-\infty}^{\infty} \phi(n) e^{inx} dn$$

taken along the real axis is identical with the integral of the same function taken about the path shown in Fig. 3, provided: (a) the function $\phi(n)$ is essentially singular at infinity; (b) the essential singularities of ϕ can be joined by a set of cuts which nowhere intersect the path of integration; and (c) the radius N of the circular part of the path is larger than the modulus of n for that singular point of ϕ which is farthest removed from the origin.

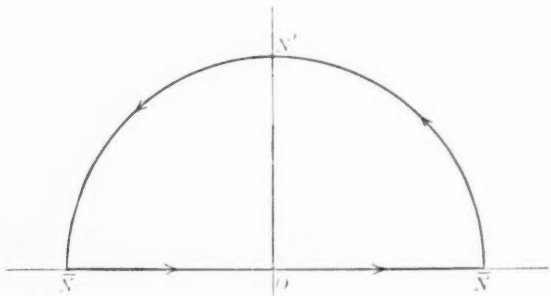

FIG. 3.

Of course, if $x < 0$, the point N' must be taken below instead of above the real axis, while if $x = 0$ the argument fails altogether.

6. **Differentiation and integration under the sign of integration.** In every case except $x = 0$ the divergent integral (4) has been evaluated in the Cæsaro sense, and the result has been proved to be identical with a Cauchy integral in the complex plane. This opens up the possibility of applying much of the elementary theory of analytical functions to the particular type of divergent integrals with which this paper is concerned.

It is not intended, however, to follow out this line of development, except to the extent of noting a few of those facts regarding differentiation and integration under the integral sign which will be needed in the work which follows.

The possibility of expanding $\phi(n)$ in an absolutely and uniformly convergent series of the form $\phi(n) = \sum_{j=-\nu}^{\infty} a_j/n^j$ has been assumed. Consider now the series which results upon multiplying both sides of this equation by n^k; k being a positive integer. This series,

$$\phi_k(n) = n^k\phi(n) = \sum_{j=-\nu}^{\infty} \frac{a_j}{n^{j-k}},$$

which again converges uniformly and absolutely for $n \geqq N$, satisfies all the conditions imposed upon ϕ; so that the integral of ϕ_k from $-\infty$ to $+\infty$ may be evaluated about the same closed path as that used for ϕ. In the Cauchy form, however

$$i^k \int \phi_k(n)e^{inx}dn = \int \phi(n)\frac{\partial^k}{\partial x^k}e^{inx}dn = \frac{\partial^k}{\partial x^k}\int \phi(n)e^{inx}dn,$$

since there can be no question as to the propriety of differentiating under the sign of integration along the closed path. Owing to the equivalence of this path with the real axis it follows that

$$\frac{\partial^k}{\partial x^k}\int_{-\infty}^{\infty} \phi(n)e^{inx}dn = \int_{-\infty}^{\infty} \phi(n)\frac{\partial^k}{\partial x^k}e^{inx}dn.$$

THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, it is permissible to differentiate the integral* (4) *repeatedly with respect to x under the sign of integration.*

In case the function

$$\Phi_1(x) = \int_{-\infty}^{\infty} \frac{\phi(n)}{in}(e^{inx} - e^{in\mu})dn$$

is considered and the above theorem applied, there results the equation

$$\Phi(x) = \frac{\partial \Phi_1}{\partial x} = \int_{-\infty}^{\infty} \phi(n)e^{inx}dn,$$

which is true for all values of x different from zero. That is $\Phi_1(x)$ is a primitive of $\Phi(x)$ which vanishes when $x = \mu$; therefore

$$\int_{\mu}^{x} dx \int_{-\infty}^{\infty} \phi(n)e^{inx}dn = \int_{-\infty}^{\infty} dn \int_{\mu}^{x} \phi(n)e^{inx}dx. \qquad (16)$$

This argument breaks down if the signs of μ and x are not alike, since in this case the x-integration passes over the value $x = 0$ for which the

integral (4) can not be reduced to a closed path. In this case $\Phi(0)$ will not ordinarily possess a meaning and therefore it is impossible to assign a meaning to (16). In the special case in which $\Phi(x)$ has a meaning even when $x = 0$, this objection no longer applies and the equation (16) is still true.*

THEOREM 3. *If the conditions of Theorem 1 are satisfied, the integral (4) may be integrated with respect to x under the sign of integration, between limits of like sign.*

7. The Fourier integral identity: the function $\Psi(\lambda, t)$. The ideas presented above throw a rather interesting light upon the Fourier integral identity, and inasmuch as a consideration of this identity will also serve the purpose of introducing a certain function $\Psi(\lambda, t)$ which will be needed in a number of places later on, it may not be amiss to give it consideration at this time.

Consider the function

$$\Psi(\lambda, t) = i \int_{-\infty}^{\infty} \frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\, dn, \qquad (\mu \neq 0,\ t \neq \lambda). \tag{17}$$

It is Riemann integrable, if $t \neq \lambda$; but what is of more immediate consequence it may be evaluated as a Cauchy integral, as explained in section 5. Its value is thus found to be†

$$\Psi(\lambda, t) = 0, \qquad \lambda < t; \qquad \Psi(\lambda, t) = 2\pi, \qquad \lambda > t.$$

When $t = \lambda$ it is indeterminate. This uncertainty may be overcome by means of the definition

$$\Psi(\lambda, t) = 0, \qquad \lambda < t; \qquad \Psi(\lambda, t) = 2\pi, \qquad \lambda \geqq t. \tag{18}$$

Hereafter, throughout the entire paper, the notation $\Psi(\lambda, t)$ will be consistently used to refer to the function defined by (18); and will only incidentally have any relation to equation (17).

Now build up the Stieltjes integral $\int_{-\infty}^{\infty} f(\lambda) d\Psi(\lambda, t)$. Quite obviously this integral has the value $2\pi f(t)$, whatever t may be, so that it is possible to state at once the identity

$$2\pi f(t) \equiv \int_{-\infty}^{\infty} f(\lambda) d\Psi(\lambda, t). \tag{19}$$

Returning again to the consideration of (17), it is to be noted that $\partial\Psi/\partial\lambda$ may be obtained by differentiating (17) under the sign of integra-

* It is easily seen that in case $\Phi(x)$ is defined for $x = 0$, all of the integrals involved in the above discussion are convergent in the Riemann sense and there is therefore no necessity of using the Cæsaro definition.

† These values are computed on the assumption that $\mu > 0$, as it may perfectly well be.

tion, except when $\lambda = t$. Performing this differentiation, it is found that $d\Psi = d\lambda \int_{-\infty}^{\infty} e^{in(t-\lambda)}dn$, for $\lambda \neq t$. When $\lambda = t$, the right-hand side of this equation is meaningless,—although there is no reason why it should not be given a meaning by definition. At present, however, this will not be done, and only formal equations will be dealt in.

Substituting the formal value of $d\Psi$ in equation (19), the result

$$2\pi f(t) = \int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} e^{in(t-\lambda)}dn \tag{20}$$

is obtained. This is the Fourier identity as ordinarily written; and the derivation given reveals its formal equivalence with the rigorous identity (19). In what follows, (19) may occasionally be termed "Fourier's Integral"; (20) being regarded as a meaningless formal equivalent, which, like derivative notation, claims recognition because of its universal familiarity.

Finally, it should be noted that $\int_a^b f(\lambda)d\Psi(\lambda, t)$ is equivalent to (19), provided $a < t \leqq b$. In case $a \geqq t$ or $t > b$ this equivalence no longer obtains.

8. **The integral**

$$G(t) = \int_a^b \chi(\lambda, t)f(\lambda)d\lambda. \tag{21}$$

In interpreting the divergent Fourier identity in the last paragraph, a very simple Stieltjes integral was found of value. The next few sections will be devoted to developing in a simple manner such of the properties of these integrals as will be of service later in assigning a meaning to a broad class of divergent integrals.

Let $f(\lambda)$ be a function of bounded variation within the closed interval (ab); and let $\chi_1(\lambda, t)$ and $\chi_2(\lambda, t)$ be two functions which are regular in λ throughout the entire interval (ab) for values of t in a certain interval (t_1, t_2) which includes (a, b). It will be assumed in the following that t takes no values outside (a, b). Furthermore, let

$$\chi_1(t, t) = \chi_2(t, t),$$

and define the function

$$\chi(\lambda, t) = \chi_1(\lambda, t) \qquad (\lambda \leqq t); \qquad \chi(\lambda, t) = \chi_2(\lambda, t) \qquad (\lambda \geqq t).$$

This is the function which occurs in equation (21).

It is quite obvious that (21) becomes

$$G(t) = \int_a^t \chi_1(\lambda, t)f(\lambda)d\lambda + \int_t^b \chi_2(\lambda, t)f(\lambda)d\lambda,$$

and that each of these integrals may be differentiated under the sign of integration, so that

$$\frac{\partial G}{\partial t} = \int_a^t \frac{\partial \chi_1(\lambda, t)}{\partial t} f(\lambda) d\lambda + \int_t^b \frac{\partial \chi_2(\lambda, t)}{\partial t} f(\lambda) d\lambda. \tag{22}$$

But in the interval $(a, t, \partial\chi_1/\partial t = \partial\chi/\partial t$, while in the interval (t, b), $\partial\chi_2/\partial t = \partial\chi/\partial t$. Thus the two integrals of (22) may be included under one sign by replacing $\partial\chi_1/\partial t$ and $\partial\chi_2/\partial t$ by $\partial\chi/\partial t$, except for the fact that the expression $\partial\chi/\partial t$ has no meaning at the point $\lambda = t$. This objection is easily overcome, however, by replacing both $\partial\chi_1/\partial t$ and $\partial\chi_2/\partial t$ by a unilateral derivative of χ; thus altering the integrand of (22) by at most a finite amount at one point only. To be explicit, the left-hand derivative will be used, so that

$$\frac{\partial G}{\partial t} = \int_a^b f(\lambda) \frac{\partial \chi}{\partial t} d\lambda. \tag{23}$$

As a special case, consider the function $\chi(\lambda, t)$ defined by the equation

$$\chi(\lambda, t) = \int_{-\infty}^{\infty} \phi(n) \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn, \tag{24}$$

where it is assumed that $\phi(\infty) = 0.$[4] It is easily seen that the path of integration, which in (24) is the real axis, may be distorted into the path A of Fig. 4, provided $\phi(n)$ has no singularities infinitely near the real axis.

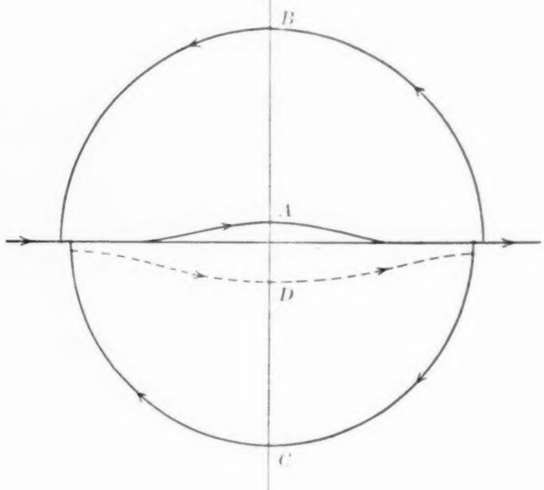

FIG. 4.

But along this path each separate term of χ is regular and may be evaluated independently. Further, each of them may be reduced to some such

path as AB or AC, according as the coefficient of n in the exponent of (24) is positive or negative.

In particular, if μ is positive, the integral

$$\int \frac{\phi(n)e^{in\mu}dn}{n} dn$$

must be taken along the path AB. It is independent of both λ and t, and may be denoted by a constant K. As for the other term,

$$\int \frac{\phi(n)e^{in(t-\lambda)}}{n} dn \tag{25}$$

it must be evaluated along the path AB when $t > \lambda$, and along the path AC when $t < \lambda$. That is, unless $t = \lambda$, χ is equal to one of the functions

$$\chi_1(\lambda, t) = -K + \int_{AB} \frac{\phi(n)e^{in(t-\lambda)}}{n} dn,$$

$$\chi_2(\lambda, t) = -K + \int_{AC} \frac{\phi(n)e^{in(t-\lambda)}}{n} dn.$$

But these functions are analytic in both λ and t, and therefore satisfy the conditions laid down at the beginning of this section. Moreover, owing to the fact that $\phi(\infty)$ has been assumed zero, it is easy to prove by an argument similar to that of section 5 that (25) may be evaluated about a closed path even when $\lambda = t$. Indeed, in this case, it does not matter which of the paths AB or AC is used, so that $\chi(t, t) = \chi_1(t, t) = \chi_2(t, t)$. This establishes the fact that χ is everywhere continuous in the interval (a, b) and therefore satisfies the conditions imposed in the discussion of equation (21).

It therefore follows immediately that (23) is satisfied by this function χ. It is also immediately seen that (23) may be written in the alternative form

$$\frac{\partial}{\partial t}\int_a^b f(\lambda)\chi(\lambda, t)d\lambda = -\int_a^b f(\lambda)d\chi(\lambda, t), \tag{26}$$

since

$$\left| \frac{\partial \chi}{\partial t} = -\frac{\partial \chi}{\partial \lambda} \right|$$

These are the formulæ for differentiating under the sign of integration. The conditions imposed should be carefully noted. They are:

With regard to ϕ, that it is regular along the real axis, and zero at infinity, and that its essential singularities, if they exist, are of such a character that the cuts of the Riemann surface on which ϕ is analytic need not intersect the real axis; and

With regard to f, that it is a function of bounded variation in (ab).

In carrying out the proof, the tacit assumption has been made that a and b do not vary with t. This assumption is not necessary, however, provided the customary terms are added to (23) and (26).

It should also be observed that the proof is in no way dependent upon the fact that $\phi(n)$ is not a function of t, and is equally true even when ϕ varies with t. In this case, however, the alternative form (26) cannot be used, since it is no longer true that

$$\frac{\partial \chi}{\partial t} = -\left.\frac{\partial \chi}{\partial \lambda}\right| .$$

9. **The Stieltjes integral,**

$$\Phi(t) = \int_a^b f(\lambda) d\chi(\lambda, t) \tag{27}$$

Differentiation under the sign of integration. Stieltjes has shown that it is permissible to integrate an integral of the type (27) by parts, provided $f(\lambda)$ is not discontinuous at $\lambda = t$ and is a function of bounded variation.* Therefore, denoting $f(b)\chi(b, t) - f(a)\chi(a, t)$ by $F(a, b, t)$,

$$\Phi(t) = F(a, b, t) - \int_a^b \chi(\lambda, t) df(\lambda).$$

Assume for simplicitly that $f(\lambda)$ is discontinuous at one point only in the open interval† a, b; and that this point is $\lambda = \Lambda$. Denote by $2\pi\delta$ the value $f(\Lambda + 0) - f(\Lambda - 0)$. The exact value of $f(\Lambda)$ is of no consequence, since it contributes nothing to $\Phi(t)$. If it is not equal to $f(\Lambda + 0)$, a new function $f(\lambda + 0)$ may be substituted for $f(\lambda)$ in (27) without in any way altering the value of Φ. It will be assumed that this substitution has been made; but for simplicity it will not be explicitly set forth in the notation.

Consider the new function $\bar{f}(\lambda)$ which is defined as

$$\bar{f}(\lambda) = f(\lambda) - \delta \cdot \Psi(\lambda, \Lambda),$$

where Ψ is defined by (18). Then $\bar{f}(\lambda)$ is continuous, even at $\lambda = \Lambda$, and $df(\lambda)$ may be replaced by

$$\delta \cdot d\Psi(\lambda, \Lambda) + \left.\frac{d\bar{f}}{d\lambda}\right| d\lambda.$$

* The conditions stated by Stieltjes were not as broad as these; but he indicates clearly that he knew they were more restrictive than necessary.

† If f is discontinuous at an endpoint, say at a, it may be replaced by the function f_1 defined as

$$f_1 = 0 \qquad (\lambda < a); \qquad\qquad f_1 = f(\lambda) \qquad (\lambda \geqq a).$$

Then $\Phi(t)$ may be written as

$$\Phi(t) = \int_{a-\eta}^{b} f_1(\lambda) d\chi(\lambda, t).$$

The function f_1 is not discontinuous at the limits of integration, and has only one discontinuity In case a is a function of t, this process is still possible. The fact that two discontinuities are introduced is obviously not essential.

That is,

$$\Phi(t) = F(a, b, t) - \int_a^b \chi(\lambda, t) \cdot \left|\frac{d\bar{f}}{d\lambda}\right| d\lambda - 2\pi\delta \cdot \chi(\Lambda, t).$$

Differentiating, and making use of formula (23), the value of $d\Phi/dt$ is obtained in the form

$$\frac{d\Phi}{dt} = \frac{dF}{dt} - 2\pi\delta\frac{d\chi(\Lambda, t)}{dt} - \int_a^b \left|\frac{d\bar{f}}{d\lambda}\right| \cdot \left|\frac{\partial\chi}{\partial t} d\lambda\right.$$
$$- \frac{db}{dt}\left.\frac{d\bar{f}(b)}{db}\right| \chi(b, t) + \frac{da}{dt}\left.\frac{d\bar{f}(a)}{da}\right| \chi(a, t).$$

This equation is capable of very considerable simplification. In the first place,

$$\left.\frac{d\bar{f}}{d\lambda}\right| = \frac{df}{d\lambda}$$

except at $\lambda = \Lambda$. But since Λ is different from both b and a, the dashes can be removed from all f's which are not under the sign of integration. Furthermore,

$$2\pi\delta\frac{d\chi(\Lambda, t)}{dt}$$

is obviously equal to

$$\int_a^b \delta \left| \frac{d\chi(\lambda, t)}{dt} d\Psi(\lambda, \Lambda); \right.$$

and may be combined with the integral which follows it. But when this is done, inspection shows that the entire integral term may be rewritten in the form

$$\int_a^b \left| \frac{\partial\chi(\lambda, t)}{\partial t} df(\lambda). \right.$$

Collecting these results, and once more applying the formula of integration by parts it is found that

$$\frac{d\Phi}{dt} = \int_a^b f(\lambda) d \left| \frac{\partial\chi(\lambda, t)}{dt} + \frac{db}{dt} f(b) \frac{\partial\chi(b, t)}{\partial b} - \frac{da}{dt} f(a) \frac{\partial\chi(a, t)}{\partial a}. \right. \qquad (28)$$

This is very similar to the ordinary formula of differentiation under the sign of integration. It is quite obvious that it may be extended to the case where $f(\lambda)$ has a finite number of discontinuities.

Since there is no distinction between $\partial\chi/\partial t$ and $|\,\partial\chi/\partial t$ except at the point $\lambda = t$, the value of $|\,\partial\chi/\partial t$ may be found by evaluating the formal derivative of (24),

$$\left|\frac{\partial\chi}{\partial t} = -\int e^{in(t-\lambda)}\phi(n)dn, \right. \qquad (29)$$

about one of the closed paths AB and AC. This is immediately obvious when $\lambda \neq t$. But it has already been shown that $\chi(\lambda, t) = \chi_2(\lambda, t)$ when $t \leqq \lambda$, and that this function χ_2 is analytic in both λ and t. Hence it follows that at $\lambda = t$,

$$\frac{\partial \chi}{\partial t} = \frac{\partial \chi_2}{\partial t};$$

which is the integral (29) evaluated about the path AB. Thus the propriety of differentiating (27) under both signs of integration is established and the following theorem is obtained:

THEOREM 4. *Let $f(\lambda)$ be a function of bounded variation in (a, b), which is not discontinuous at $\lambda = t$. Let $\phi(n)$ be regular along the real axis, zero at infinity, and capable of representation upon a Riemann's surface in such a way that a point traversing the entire real axis remains continually on the same sheet. Define $\chi(\lambda, t)$ as in (24), and $\Phi(t)$ as in (27). Then the derivative $d\Phi/dt$ exists, provided db/dt and da/dt do, and may be obtained by differentiation under the sign of integration in accordance with the formula (28), in which*

$$\frac{\partial \chi}{\partial t} = -\int e^{in(t-\lambda)}\phi(n)dn,$$

taken about AB if $t > \lambda$, otherwise about AC.

In the application to physical problems, there is no occasion to use variable limits; in consequence of which b and a will be assumed constant in what follows.

The result expressed by (28) may be extended to the case of infinite limits of integration. Assume that $f(\lambda)$ has, in the interval (a, ∞) only a finite number of discontinuities and that no point on the real axis is either a singular point or a limiting point of the singularities of $\phi(n)$. There is then a finite number b so large that in the interval (b, ∞) $f(\lambda)$ is continuous. It will be assumed that b is chosen large enough to satisfy the condition $b > t$, for all values of t which need be considered.

Then

$$\Phi_1(t) = \int_a^\infty f(\lambda)d\chi(\lambda, t) = \int_a^b f(\lambda)d\chi(\lambda, t) + \int_b^\infty f(\lambda)\frac{\partial \chi}{\partial \lambda}d\lambda.$$

The former of these integrals comes under the proof already given, and only the latter need be considered. This, however, is an ordinary Riemann integral, and may be differentiated repeatedly under the sign of integration *so long as the resultant integrals are uniformly convergent*, as they will presently be proved to be.

For all values of $\lambda > t$, and hence for all values of $\lambda \geqq b$, $\chi(\lambda, t)$ may

be found by evaluating

$$\int \phi(n)\frac{e^{in(t-\lambda)}}{n}dn$$

about the path AC, and

$$\int \phi(n)\frac{e^{in\mu}}{n}dn$$

about the path AB. The latter of these, so far as either λ or t is concerned, is a constant, which has already been denoted by K. The former, which may be called $F(\lambda, t)$ may readily be reduced to the form

$$F(\lambda, t) = -2\pi i\phi(0) + \int_{DC}\frac{\phi(n)}{n}e^{in(t-\lambda)}dn,$$

the path DC (shown in Fig. 4), being at no point removed less than a finite distance η from the real axis. Hence

$$\chi(\lambda, t) = -K - 2\pi i\phi(0) + \int_{DC}\frac{\phi(n)}{n}e^{in(t-\lambda)}dn,$$

and, whatever the values of λ and t,

$$\frac{\partial^{j+1}\chi}{\partial t^j\partial\lambda} = -i^{j+1}\int_{DC}n^j\phi(n)e^{in(t-\lambda)}dn.$$

The path DC possesses a finite length L. Along it the value of n is constantly finite and less than N; $\phi(n)$ is constantly less than a maximum value M, and

$$|e^{in(t-\lambda)}| \leqq e^{-\eta(\lambda-t)} \leqq e^{-\eta(\lambda-b)} \qquad (\lambda \geqq b \geqq t).$$

Thus

$$\left|\frac{\partial^{j+1}\chi}{\partial t^j\partial\lambda}\right| \leqq MLN^je^{-\eta(\lambda-b)} \qquad (\lambda \geqq b \geqq t).$$

With regard to $f(\lambda)$, which has been assumed to be a function of limited variation in (a, b), it is sufficient to assume the condition

$$|f(\lambda)| < A\cdot\lambda^q \qquad (\lambda \geqq b).$$

Then it is true that, for any s greater than b,

$$\left|\int_s^\infty f(\lambda)\frac{\partial^{j+1}\chi}{\partial t^j\partial\lambda}d\lambda\right| \leqq ALMN^je^{\eta b}\int_s^\infty \lambda^qe^{-\eta\lambda}d\lambda \leqq ALMN^je^{-\eta(s-b)}P(s),$$

where $P(s)$ is a polynomial of degree q in s. This evidently converges to zero uniformly in t as s is increased indefinitely. *Thus the applicability of (28) to infinite limits is established if $f(\lambda)$ possesses only a finite number of discontinuities; if $|f(\lambda)| < A\cdot\lambda^q$, $\lambda \geqq b$; and if the singularities of $\phi(n)$ have no limiting points upon the real axis.*

Finally it should be noted that (28) is still true when ϕ is a function of both n and t. In this case the form of (29) is altered, since the equation as written no longer represents the result of differentiating (24) under the sign of integration.

10. **The Stieltjes integral (27): inversion of integrals.** Consider the two functions

$$\chi(\lambda, t) = i\int_{-\infty}^{\infty} \phi(n) \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn$$

and

$$\chi_1(\lambda, t) = \int_{-\infty}^{\infty} \phi(n) \frac{e^{in(t-\lambda)} - e^{in(\nu-\lambda)} - ine^{in\mu}(t-\nu)}{n^2} dn,$$

where $\phi(n)$ at infinity is regular, but not necessarily zero. The integrand of χ_1 has no singularities upon the real axis; and its path of integration may therefore be distorted into the path A of Fig. 4. This having been done, χ_1 may be rewritten in the form

$$\chi_1(\lambda, t) = \int_A \frac{\phi(n)}{n} \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn - \int_A \frac{\phi(n)}{n} \frac{e^{in(\nu-\lambda)} - e^{in\mu}}{n} dn$$
$$- i(t-\nu)\int_A \frac{\phi(n)}{n} e^{in\mu} dn.$$

The third of these terms may be evaluated about the path AB, regardless of the values of t and λ. The second integral can be evaluated about either the path AB or the path AC, and is independent of t. The first term is of the type discussed in section 9, except that the $\phi(n)$ of section 9 corresponds to the $\phi(n)/n$ of this section.* Hence it is immediately obvious that the entire argument of section 9 applies equally well to the expression $\Phi_1(t)$ defined by the equation

$$\Phi_1(t) = \int_a f(\lambda) d\chi_1(\lambda, t).$$

In particular,

$$\frac{\partial \Phi_1}{\partial t} = \int_a^b f(\lambda) d\left|\frac{\partial \chi_1(\lambda, t)}{\partial t}\right|,$$

$\dfrac{\partial \chi_1(\lambda, t)}{\partial t}$ representing the left-hand unilateral derivative of χ.

But $\left|\dfrac{\partial \chi_1}{\partial t}\right|$ may be found by actual computation to be, equal to $\chi(\lambda, t)$, except possibly for the value $\lambda = t$; so that

$$d\left|\frac{\partial \chi_1}{\partial t}\right| = d\chi.$$

* The $\phi(n)$ of section 9 was required to have no singularities on the real axis. This condition was necessary to justify the shift from the real axis to the path A as a path of integration, and was thereafter of no consequence. This shift having already been accomplished, the fact that $\phi(n)/n$ is not necessarily regular at $n = 0$ causes no difficulty.

Therefore, if

$$\Phi(t) = \int_a^b f(\lambda) d\chi(\lambda, t),$$

it may be said that

$$\frac{\partial \Phi_1}{\partial t} = \Phi.$$

Taking account of the fact that $\Phi_1 = 0$ when $t = \nu$, it follows that

$$\Phi_1 = \int_\nu^t \Phi dt. \tag{30}$$

But, quite obviously, $\chi_1(\lambda, t)$ is the result of integrating $\chi(\lambda, t)$ with respect to t between the limits ν and t; the integration being performed *under* the sign of integration. That is (30) expresses the legitimacy of integration under both signs of integration.

11. **The Ψ-functions.** Up to the present point in this discussion no use has been made of the theory of divergent integrals as developed in the early sections. It will be the purpose of this and the remaining sections to apply this theory to the Stieltjes integral, and to the solution of differential equations. For generality, it is assumed that ϕ is a function of both n and t.

Let there be a function $\phi(n, t)$ which vanishes at infinity, and is regular over a certain range of values of t for sufficiently large values of n. Let

$$\chi(\lambda, t) = \int_{-\infty}^{\infty} \phi(n, t) \frac{e^{in(t-\lambda)} - e^{in\mu}}{n} dn$$

and let

$$\Phi(t) = \int_a^b f(\lambda) d\chi(\lambda, t).$$

It has been shown that this function Φ may be differentiated under both signs of integration, even when $\lambda = t$, provided that the resulting $\left|\frac{\partial \chi}{\partial t}\right|$ is evaluated about the path AB if $t > \lambda$, and about the path AC if $t \leqq \lambda$. This is true, however, only because χ is continuous, and if $\left|\frac{\partial \chi}{\partial t}\right|$ is discontinuous $d^2\Phi/dt^2$ cannot be found in this way. It is therefore necessary to consider the magnitude of the discontinuity of $\left|\frac{\partial \chi}{\partial t}\right|$, and its effect upon the second derivative of Φ.

The conditions imposed upon ϕ justify the expansion

$$\phi(n, t) = \sum_{j=1}^{\infty} \frac{a_j(t)}{n^j};$$

which is term by term differentiable with respect to t. Making use of this form of expansion it may be said that

$$\frac{\partial^\sigma}{\partial t^\sigma}\left[\phi(n, t)\frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\right] = -\frac{e^{in\mu}}{n}\frac{\partial^\sigma \phi}{\partial t^\sigma} + \sum_{j=1}^{\sigma}\sum_{k=0}^{\sigma} C_k^\sigma i^k n^{k-j-1}\frac{d^{\sigma-k}a_j}{dt^{\sigma-k}}e^{in(t-\lambda)}.$$

Some of the terms in this expansion involve powers of n higher than -2. Denote their sum by P_σ. Then

$$P_\sigma = \sum_{j=1}^{\sigma}\sum_{k=j}^{\sigma} C_k^\sigma i^k n^{j-2}\frac{d^{\sigma-k}a_{k-j+1}}{dt^{\sigma-k}}e^{in(t-\lambda)}.$$

Finally, the functions ϕ_σ and χ_σ are defined as

$$\phi_\sigma(n, t) = \frac{\partial^\sigma}{\partial t^\sigma}\left[\phi\frac{e^{in(t-\lambda)} - e^{in\mu}}{n}\right] - P_\sigma, \qquad \chi_\sigma(\lambda, t) = \int \phi_\sigma(n, t)dn,$$

the integral in the last equation being evaluated about AB if $\lambda < t$; otherwise about AC.

It is seen at once that every ϕ_σ is of degree -2 in n. Therefore every χ_σ is continuous* at $\lambda = t$, and every integral of the type $\int_a^b f(\lambda)d\chi_\sigma$ may be differentiated with respect to t under the integral signs. But by actual differentiation

$$\frac{d\phi_\sigma}{dt} = \phi_{\sigma+1} + \frac{i}{n}e^{in(t-\lambda)}\left[Q_{\sigma+1} - \frac{dQ_\sigma}{dt}\right],$$

where

$$Q_\sigma(t) = \sum_{k=1}^{\sigma} C_k^\sigma i^{k-1}\frac{d^{\sigma-k}a_k}{dt^{\sigma-k}}.$$

Hence†

$$\frac{d\chi_\sigma(\lambda, t)}{dt} = \chi_{\sigma+1} + \Psi(\lambda, t)\left[Q_{\sigma+1} - \frac{dQ_\sigma}{dt}\right]$$

and

$$\frac{\partial\Phi}{\partial t} = \int_a^b f(\lambda)d\,\frac{\partial\chi}{\partial t} = \int_a^b f(\lambda)d(\chi_1 + a_1\Psi) = 2\pi a_1(t)f(t) + \int_a^b f(\lambda)d\chi_1(\lambda, t).$$

Having once obtained this formula, it is easy to establish by induction the following theorem:

THEOREM 5. *Let $\phi(n, t)$ be a function which, for a certain range of values of t to which attention is confined, and for n real or sufficiently large, is regular in both t and n, and which vanishes at $n = \infty$. Then the Stieltjes Integral $\Phi(t)$ possesses a σ'th derivative provided $f(t)$ possesses a $\sigma - 1$'th*

* See footnote on page 33.

† The identification of $-\int \frac{e^{in(t-\lambda)}}{in}dn$ with $\Psi(\lambda, t)$ is established by direct evaluation.

derivative; and this derivative may be obtained by differentiating the Stieltjes integral Φ *under all signs of integration, according to the formula*

$$\frac{\partial^\sigma \Phi}{\partial t^\sigma} = 2\pi\left[\frac{\partial^{\sigma-1}(a_1 f)}{\partial t^{\sigma-1}} + \sum_{j=0}^{\sigma-2}\frac{\partial^j}{\partial t^j}\left(fQ_{\sigma-j} - f\frac{\partial Q_{\sigma-j-1}}{\partial t}\right)\right. \\ \left. + \int_a^b f(\lambda)d\chi_\sigma(\lambda, t)\right]. \tag{31}$$

Equation (31) gives the true value of the derivatives of Φ, by introducing into the formal derivative certain auxiliary terms. For purposes of comparison it is desirable to write down the result of differentiating $\Phi(t)$ formally under all signs of integration. The result is

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)d\chi^{(\sigma)}, \tag{32}$$

where

$$\chi^{(\sigma)}(\lambda, t) = \int_{-\infty}^{\infty}[\phi_\sigma(n, t) + P_\sigma(n, t)]dn. \tag{33}$$

It has been shown that this *formal* result for $\chi^{(\sigma)}$ may be evaluated about one of the paths AB or AC, *so long as* $t \neq \lambda$. What is not known is *that when the result is substituted in* (32) *it results in* $\partial^\sigma\Phi/\partial t^\sigma$; and indeed, so long as $d\chi^{(\sigma)}$ does not have a meaning at $\lambda = t$, the statement in italics is absurd. Ignoring this difficulty for the moment, consider the actual value of $\chi^{(\sigma)}$ for $t \neq \lambda$. Only those terms of P_σ for which the exponent of n is -1 contribute anything to the result. Hence, separating out these terms it is seen that

$$\int_{-\infty}^{\infty} P_\sigma(n, t)dn = \Psi(\lambda, t)Q_\sigma(t),$$

and therefore

$$d\chi^{(\sigma)}(\lambda, t) = d\chi_\sigma(\lambda, t) + Q_\sigma(t)d\Psi(\lambda, t) = d\chi_\sigma(\lambda, t) \qquad (\lambda \neq t). \tag{34}$$

This is exactly the function with respect to which the integral term of (31) is integrated for all values of λ.

The first observation to which this equality leads is concerned with the result of a substitution of (34) in (31). Suppose (33) to be evaluated about AC when $\lambda = t$; that is, suppose the same convention regarding the path of integration is made in (33) as in (31). Then (34), in its first form, is true even when $\lambda = t$, and therefore *it is not true that*

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)d\left|\frac{\partial^\sigma \chi}{\partial t^\sigma}\right.,$$

in the general case. In fact, the deficiency of this equation is exactly the difference of the group of terms which do not occur under the sign of integration in (31), *and the term* $2\pi f(t)Q_\sigma(t)$.

The second observation concerns those terms of P_σ which contribute nothing to the value of (33) when $\lambda \neq t$. *Formally* the corresponding terms of (33) are the successive derivatives of the equation (17), multiplied by a factor independent of n. *Formally*, therefore, (32) becomes

$$\frac{d^\sigma \Phi}{dt^\sigma} = \int_a^b f(\lambda)\left[d\chi_\sigma + \sum_{j=1}^{\sigma}\sum_{k=j}^{\sigma} C_k^\sigma \frac{d^{\sigma-k}a_{k-j+1}}{dt^{\sigma-k}} i^{k-j} d\Psi^{(j-1)}\right].$$

This result is significant, for if the definitions

$$\int_a^b f(\lambda)d\Psi^{(\sigma)}(\lambda, t) = 2\pi f^{(\sigma)}(t), \qquad (35)$$

which are suggested by the *formal* differentiation of (19), are made for *all* values of σ, it follows that (32) is a true equation.*

The definitions (35), however, are sufficient to give a meaning to an integral of the type (32) regardless of how it may have been derived—whether by differentiation or otherwise—and are therefore of very general importance. In fact, it requires but a moment's reflection to observe the truth of the following statements:

DEFINITION 1. *Let $\Psi^{(\sigma)}(\lambda, t)$ represent the divergent integral*

$$i^{\sigma+1}\int_{-\infty}^{\infty} n^{\sigma-1}e^{in(t-\lambda)}dn.$$

Then the identity

$$2\pi f^{(\sigma)}(t) \equiv \int_a^b f(\lambda)d\Psi^{(\sigma)}(\lambda, t)$$

defines the integral in its right-hand member, provided $a < t < b$.

DEFINITION 2. *Let $\phi(n)$ be a function which, for a certain range of values of t to which attention is confined, and for real values of n, is regular in both n and t; and which may be expanded in a series*

$$\phi(n, t) = \sum_{j=-\nu}^{\infty}\frac{a_j(t)}{n^j},$$

convergent for sufficiently large values of n. Also let $\chi(\lambda, t)$ be defined by the equation

$$\chi(\lambda, t) = \int_{-\infty}^{\infty}\phi(n, t)e^{in(t-\lambda)}d\lambda.$$

* This is shown, of course, by establishing that the equation preceding (35) is equivalent to (31). The algebraic manipulation involved is somewhat simplified by the use of the identities:

$$(a)\ C_k^\sigma = C_k^{\sigma-1} + C_{k-1}^{\sigma-1}, \qquad (b)\ C_k^\sigma = \sum_{j=1}^{\sigma-k} C_{k-1}^{\sigma-1}, \qquad (c)\ C_{\alpha+\beta+1}^{\sigma+1} = \sum_{j=\beta}^{\sigma-\alpha} C_\beta^j C_\alpha^{\sigma-j}.$$

Those who are not familiar with these identities may establish (a) by direct addition; (b) by repeated application of (a); and (c) by induction, assuming it to hold for all values of σ and β when $\alpha = \alpha$ and showing that this involves its validity for $\alpha = \alpha + 1$. In this process $C_\alpha^{\sigma-j}$ is replaced by the summation (b); and it is noted that when $\alpha = 0$, (c) reduces to (b).

Then the identity

$$\int_a^b f(\lambda)d\chi(\lambda, t) \equiv -2\pi \sum_{j=-v}^{0} i^j a_j(t) f^{(1-j)}(t) + \int_a^b f(\lambda)d\chi(\lambda + 0, t)$$

defines the integral in its left-hand member, provided $a < t < b$.

THEOREM 6. *The definitions 1 and 2 are consistent with the operations of addition, differentiation, integration, and multiplication by a constant when these operations are performed under the sign of integration. Furthermore, if by any of these processes an integral is derived which may be evaluated as an iterated improper Riemann integral, the value of this Riemann integral is consistent with the value obtained by performing the same operations upon*

$$\Phi(t) = \int_a^b f(\lambda)d\chi(\lambda, t).$$

Not all of the details of proof of this general result are to be found in the preceding sections. It may be well, therefore, to rapidly indicate them in this place.

In the matter of addition and multiplication by a constant there are no difficulties which need be considered.

As for differentiation, the difficulty is only slightly greater. For if

$$\Phi(t) = \int_a^b f(\lambda)d\chi(\lambda, t)$$

is a true equation when interpreted as above explained, it is possible to separate out all the terms of $\phi(n, t)$ of degree greater than -2 in n. These having been removed from the integral, the remaining terms of the integrand satisfy the conditions of the proof of Theorem 5, and repeated differentiation of Φ is permissible. The equivalence of the result thus derived with what would have been obtained had these terms not been removed is easily established.

The case of integration is made to depend upon that of differentiation. For if $\phi(n, t)$ satisfies the conditions of definition 2, the same is true of the function $\phi_1(n, t)$ defined by the equation

$$\phi_1(n, t) = e^{-int} \int_\mu^t e^{int}\phi(n, t)dn;$$

the path of integration being the real axis. Then the function

$$\Phi_1(t) = \int_a^b f(\lambda)d\chi_1(\lambda, t),$$

where

$$\chi_1(\lambda, t) = \int_{-\infty}^{\infty} \phi_1(n, t)e^{in(t-\lambda)}dn,$$

may be differentiated, the result of course, being the $\Phi(t)$ used above. Hence

$$\Phi_1(t) = \int_\mu^t \Phi(t)dt + C;$$

and it is easily seen that C must be zero, since $\Phi_1(\mu) = 0$. Since χ_1 is the result obtained by formally integrating χ from μ to t under the sign of integration, the desired property of integrability is established.

That the second sentence of the theorem is true may be verified from the fact that, if the conditions on $\phi(n)$ are satisfied, $d\chi$ cannot be Riemann integrable unless $\phi(\infty)$ is zero to the second order. But if by the use of the operations of addition, multiplication, differentiation and integration a function ϕ is built up which satisfies this condition, its expansion in descending powers of n will involve no terms which require the use of definition 2. The result will be found by altering the path of integration, a method which in this case is consistent with both Riemann integration and integration by the method here explained. Hence the last sentence of the theorem is true.

Theorem 6 is the principal result of this paper. It gives a meaning to many expressions which, so far as the writer is aware, have never been interpreted before. That many of these expressions could be manipulated successfully, provided the form in which they were left by the last transformation could be interpreted by ordinary means has been generally recognized by applied mathematicians; but their use was always considered as belonging to the class of questionable operations, and whenever it was found possible to do so, the results obtained were subjected to independent checks. It now appears, not only that reliance can be placed in the final results, even when they are not interpretable by the customary means, but also that each separate step of the transformations is rigorously justifiable.

It is not within the scope of this paper to enumerate the uses which may be made of the above theorems; but it may not be amiss to give one or two examples of their application to the solution of differential equations. These examples are drawn from the paper on "The Solution of Circuit Problems" to which reference has been made in the introduction, where they are treated in detail, although, of course, with a minimum of theoretical mathematics. They will be given here as briefly as possible, the reader being referred to the original article for technical elaboration.

The first of these problems, which is given in section 12, deals with an extremely simple electrical circuit, and is chosen for its didactic value only. The second, which is much more difficult, deals with the solution

of the set of partial differential equations controlling the propagation of electricity along a pair of parallel conductors. The third problem is equivalent to the solution of a set of linear differential equations of arbitrary degree. It is discussed in section 14.

12. **Example of the foregoing theory.** The flow of electricity in a circuit containing a resistance and capacity in series is in accordance with the equation

$$RI + K\int_{-\infty}^{t} Idt = E(t), \tag{36}$$

where I is the current flowing, E is the applied electromotive force, and R and K are the resistance and stiffness (reciprocal of capacity) of the circuit.

If E is a periodic function, e^{int}, then I is periodic of the same period, and the solution of the equation is*

$$I = \frac{in}{Rin + K}E(t).$$

Any form of E may be expressed by the equation

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dne^{int}\int_{-\infty}^{\infty} d\lambda E(\lambda)e^{-in\lambda},$$

which represents, when physically interpreted, a summation of periodic terms, each multiplied by an amplitude factor

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} E(\lambda)e^{-in\lambda}d\lambda.$$

It is natural, therefore, to expect a solution for I in the form of a summation of these same terms, modified by the factor $in/(Rin + K)$. The tentative result therefore is

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} dn\int_{-\infty}^{\infty} d\lambda E(\lambda)e^{in(t-\lambda)}\frac{in}{Rin + K}.$$

That this is the true solution is found by substitution in the equation to be solved.

The integration with respect to n is carried out along the path AB if $\lambda < t$; otherwise along AC. Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{in(t-\lambda)}\frac{in}{Rin + K}dn = -\frac{K}{R^2}e^{-\frac{K}{R}(t-\lambda)},$$

* The evaluation of $\int_{-\infty}^{t} Idt$ requires the use of the Cæsaro value of an improper integral.

if $\lambda < t$. Otherwise it is zero. The result for I is therefore apparently

$$I = -\frac{K}{R^2}\int_{-\infty}^{t} E(\lambda)e^{-\frac{K}{R}(t-\lambda)}d\lambda.$$

But that this result is deficient follows from the fact that there is a term of zero degree in the expansion of $in/(Rin + K)$ in descending powers of n. To this term corresponds the supplementary I term E/R, so that finally

$$I = \frac{1}{R}E(t) - \frac{K}{R^2}\int_{-\infty}^{t} E(\lambda)e^{-\frac{K}{R}(t-\lambda)}d\lambda.$$

For instance, if $E(t) = 0$ for $t < 0$ and $E(t) = 1$ for $t > 0$, then I is zero unless $t > 0$, in which case

$$I = \frac{1}{R} - \frac{K}{R^2}\int_{0}^{t} e^{-\frac{K}{R}(t-\lambda)}d\lambda = \frac{1}{R}e^{-\frac{K}{R}t}.$$

That this result is correct is capable of direct verification by substitution in the original equation. *There is no need for this,* however, except as a check upon the computation, *for the method has been thoroughly established.*

This simple problem may be solved in easier ways; but it serves the purpose in this place of showing how the definitions which have been given above overcome the deficiencies in an otherwise powerful method. A more formidable example occurs in the next section.

13. **Application to the solution of differential equations: the telegraph equation.** As stated in the introduction, the writer conceived the ideas which have formed the material for this paper from a consideration of the problem of the telegraph equation. This problem is here presented as briefly as possible, and serves to indicate the application of the Fourier integral to the solution of partial differential equations.

The propagation of electricity along a pair of parallel conducting wires takes place in accordance with a system of differential equations, which, if units are properly chosen, reduce to

$$-\frac{\partial E}{\partial x} = \frac{\partial I}{\partial t} + I, \qquad -\frac{\partial I}{\partial x} = \frac{\partial E}{\partial t} + kE. \tag{37}$$

If both I and E are assumed to be periodic in t, a solution is readily found in the form

$$E(x, t) = e^{int}[Q_1e^{imx} + Q_2e^{-imx}],$$

$$I(x, t) = -e^{int}\frac{im}{in+1}[Q_1e^{imx} - Q_2e^{-imx}],$$

$$im = \sqrt{(in+1)(in+k)}.$$

On the other hand, if E is not periodic, it can, for a given value of x, say $x = 0$, be expanded in a Fourier integral

$$E(0, t) = \int_{-\infty}^{\infty} e^{int}F(n)dn,$$

which represents a summation of such terms as the periodic one above. It is not unnatural, therefore, to expect a solution for E in the form

$$E(x, t) = \int_{-\infty}^{\infty} [Q_1(n)e^{imx} + Q_2(n)e^{-imx}]e^{int}dn.$$

If this is postulated, it is seen that

$$Q_1 + Q_2 = F = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)e^{-in} \, d\lambda,$$

wherefore, introducing the notation

$$\frac{q_i(n)}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)e^{-in\lambda}d\lambda = Q_i(n),$$

it follows that

$$E(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} E(0, \lambda)e^{in(t-\lambda)}[q_1(n)e^{imx} + q_2(n)e^{-imx}]d\lambda.$$

Inverting the order of integration, and writing

$$\chi(\lambda, t) = i\int_{-\infty}^{\infty} \frac{e^{in(t-\lambda)} - e^{in(t-\mu)}}{n} [q_1(n)e^{imx} + q_2(n)e^{-imx}]dn \tag{38}$$

this takes the form

$$E(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0, \lambda)d\chi(\lambda, t).$$

It may be true that $q_1(n)e^{imx}$ and $q_2(n)e^{-imx}$ are essentially singular at infinity, owing to the exponential factors; but this complication will cause little difficulty. As to the location of the other singularities nothing can be affirmed in general from purely mathematical considerations. They depend upon the boundary conditions of the problem, and may conceivably lie either upon, above or below the real axis. Physical considerations show, however, that in all actual cases they must lie in the upper half plane, except when the problem has been idealized to such an extent as to put some of them upon the real axis. Even in this latter case, the same physical considerations warrant the use of a path of integration sufficiently far below the real axis to avoid these singularities.

For the purpose of this argument it will be assumed that all points of the real axis are regular points, and that the cuts are fortuitously dis-

tributed, as required throughout the preceding sections. It is then possible to vary the path of integration to such an extent that it will avoid the origin, and thereafter to split (38) into four portions involving one exponential each. Consider only the first of these,

$$\chi_1(\lambda, t) = i \int_A e^{in(t-\lambda)+imx} \frac{q_1(n)}{n} dn, \tag{39}$$

which is representative of all. It is seen that, at infinity, im is approximately equal to* $-in + [(k+1)/2]$, so that the quantity $im - in$ is finite and regular there. Indeed, it is regular at all points of the complex plane, excepting the two winding points $n = i$ and $n = ki$. This having been established, (39) may be thrown into the form

$$\chi_1(\lambda, t) = i \int_A e^{in(t-\lambda-x)} \phi_1(n) dn,$$

where

$$\phi_1(n) = \frac{q_1(n)}{n} e^{i(m-n)x}.$$

This function ϕ_1 satisfies all the conditions which have been imposed upon the function ϕ throughout this paper as regards singularities. It follows that $\int_{-\infty}^{\infty} E(0, \lambda) d\chi_1$, and therefore $\int_{-\infty}^{\infty} E(0, \lambda) d\chi$, may be subjected to the processes of repeated differentiation and integration with respect to either x or t.

Suppose then that the integrals

$$\begin{aligned} E(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E(0, \lambda) d\lambda \int_{-\infty}^{\infty} e^{in(t-\lambda)} [q_1(n) e^{imx} + q_2(n) e^{-imx}] dn \\ I(x, t) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} E(0, \lambda) d\lambda \int_{-\infty}^{\infty} \frac{im}{in+1} e^{in(t-\lambda)} [q_1(n) e^{imx} - q_2(n) e^{-imx}] dn \end{aligned} \tag{40}$$

are substituted in the equations (37). Upon performing all the operations under the signs of integration, and interpreting the results according to the principles of section 11, it is seen that (37) is satisfied.

The first thought which is suggested by a glance at (40) is that a solution which appears in such a complicated form is of very little use. But it must be observed that a method of evaluation has been obtained which does not require an improper integration with respect to n—and when the complicated n—integral is removed from consideration and the expressions are written in the form $\int_{-\infty}^{\infty} E(0, \lambda) d\chi(\lambda, t)$, they *appear* extremely simple. Whatever difficulty may remain is due to the complicated

* The sign has been arbitrarily chosen to agree with the original paper.

nature of the boundary conditions, which will manifest itself in complicated values for q_1 and q_2.

The actual form which these functions take in dealing with the submarine cable, and the difficulties of numerical evaluation are discussed in the paper on "The Solution of Circuit Problems," to which attention has been invited above, and to which the reader who is interested in following the matter further is referred.

14. **Application to the solution of differential equations: the generalized Heaviside expansion.** As a further example, consider the set of differential equations,

$$\begin{aligned} &a_{11}\left(\frac{\partial}{\partial t}\right)I_1 + a_{12}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{1s}\left(\frac{\partial}{\partial t}\right)I_s = f(t),\\ &a_{21}\left(\frac{\partial}{\partial t}\right)I_1 + a_{22}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{2s}\left(\frac{\partial}{\partial t}\right)I_s = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots,\\ &a_{s1}\left(\frac{\partial}{\partial t}\right)I_1 + a_{s2}\left(\frac{\partial}{\partial t}\right)I_2 + \cdots + a_{ss}\left(\frac{\partial}{\partial t}\right)I_s = 0, \end{aligned} \tag{41}$$

in which a_{11}, a_{12}, $\cdots$, a_{ss} are linear differential operators of any order. Denote by Δ the determinant

$$\Delta = \begin{vmatrix} a_{11}(in) & a_{12}(in) & \cdots & a_{1s}(in) \\ a_{21}(in) & a_{22}(in) & \cdots & a_{2s}(in) \\ \cdot & \cdot & \cdot & \cdot \\ a_{s1}(in) & a_{s2}(in) & \cdots & a_{ss}(in) \end{vmatrix}$$

and by M_{ij} the minor of a_{ij}, and build up the expressions

$$I_j = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} \frac{M_{1j}(in)}{\Delta(in)} e^{in(t-\lambda)}dn. \tag{42}$$

If Δ has no roots for real values of n, these expressions may be repeatedly differentiated, and hence may be substituted in the set of equations (41). This being done, there result the expressions

$$Q_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \int_{-\infty}^{\infty} \frac{1}{\Delta(in)} \sum_{j=1}^{s} a_{ij}(in)M_{1j}(in)e^{in(t-\lambda)}dn.$$

But $\sum_{j=1}^{s} a_{ij}(in)M_{1j}(in)$ is equal to zero or Δ, according as i is or is not different from unity. Hence it follows that $Q_i \equiv 0$, unless $i = 1$, in which case it reduces to $f(t)$. That is (42) is a solution of the system of differential equations (41).

The functions M and Δ are polynomials in terms of in. Suppose

that M/Δ is divided out until a remainder N is obtained which is of degree lower than that of Δ; and that N/Δ is then expanded, as it may be, in a set of partial fractions. Formally the result will be*

$$\frac{M}{\Delta} = \sum_{k=0}^{K} p_k (in)^k + \sum_{k=1}^{K'} \frac{q_k}{in - in_k},$$

where K' is the degree of Δ in in. Substituting this result in (42) and evaluating the terms of the second summation by the method of section 5, and the terms of the first summation by the method of section 11, it is found that

$$I_j = \Sigma p_k \frac{\partial^k}{\partial t^k} f(t) + \Sigma q_k^+ \int_a^t f(\lambda) e^{in_k(t-\lambda)} d\lambda + \Sigma q_k^- \int_t^b f(\lambda) e^{in_k(t-\lambda)} d\lambda. \quad (43)$$

Here q_k^+ denotes one of the terms in which n_k lies in the upper half of the complex plane, and q_k^- one of the terms in which n_k lies in the lower half.

In case every p_k and q_k^- is zero, and $f(\lambda) = \Psi(\lambda)/2\pi$ this reduces to Heaviside's formula†

$$I = \sum_{k=0}^{K'} \frac{q_k}{in}(1 - e^{int}).$$

From a mathematical standpoint, the formula (43) is principally important, in that it represents a particular solution of a set of linear differential equations, written in a very explicit form. It is hardly necessary to observe that a great deal could be said about boundary conditions, and complementary solutions and the like. But these matters are of sufficient importance to justify a title of their own, and like certain others which have been mentioned in the course of the argument, must be passed by for the present.

15. **Resumé.** The paper consists of three parts, the content of which is as follows:

(*a*) A discussion of the Cæsaro limit of a class of divergent integrals leads to the conclusion that it may be found by evaluating a Cauchy integral about well-defined paths.

(*b*) It is found that an important class of Stieltjes integrals may be differentiated and integrated under the sign of integration, provided that, when differentiation is performed, left-hand unilateral derivatives are used.

* Provided the roots of Δ are all different. If any of them is repeated the form of the expansion is different.

† This formula, the importance of which in circuit theory is sufficiently attested by the fact that it possesses a name, was stated by Oliver Heaviside without proof.

(c) The results of the two preceding parts of the paper are brought to bear upon a class of double integrals which, in any ordinary sense, are without meaning. With the aid of a fortuitously chosen set of definitions of $\int_a^b f(\lambda) d\Psi^{(\sigma)}(\lambda)$, it is shown how these can be assigned meanings consistent with those operations to which they are most likely to be subjected in the solution of differential equations. Several examples illustrate the use of the theory.

The purpose underlying the choice and development of the material is not to exhaust its possibilities along any one line; but rather to indicate a general line of research which, it is hoped, will be found of sufficient practical value to justify its more extensive development.

Research Laboratory of the American Telephone and Telegraph Company and the Western Electric Company, Inc.,
November, 1918.

AN ANALYTICAL SOLUTION OF BIOT'S PROBLEM.

By Tsuruichi Hayashi.

In H. Laurent's Traité d'Analyse, tome 5, 1890, p. 110, we find the following problem due to Biot: Find a plane curve, such that all the luminous rays emanating from a fixed point, after two reflections on the curve, return to the fixed point. Laurent's solution is very simple, applying the common law of reflection, but it is wrong. Prof. M. Fijiwara has given a true solution to the problem in the Tohoku Mathematical Journal, volume 2, 1912, p. 149. I shall here give an analytical solution.

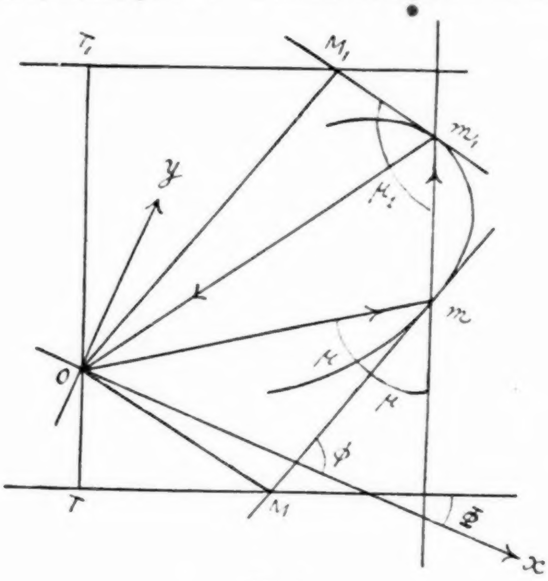

Fig. 1.

Let O be the source of light, and let Omm_1O be the path of a ray. Let μ be the angle which Om or mm_1 makes with the tangent mM at m, and let μ_1 be the angle which mm_1 or m_1O makes with the tangent m_1M_1 at m_1. Drop from O perpendiculars OM, OM_1 on mM, m_1M_1 respectively. Then M, M_1 lie on the pedal of the required curve.

Take O as origin and Ox, Oy as rectangular coördinate axes. Then the rectangular coördinates (x, y) of the point m are connected with the polar tangential coördinates (p, φ) of the same point by the relation

$$x \sin \varphi - y \cos \varphi = p,$$

μ being the perpendicular OM, and φ being the angle between Mm and Ox. This relation is the equation to the tangent Mm. Hence this equation and that got by differentiating it with respect to φ, i.e.,

$$x \cos \varphi + y \sin \varphi = dp/d\varphi, = p' \text{ say},$$

give the rectangular coördinates (x, y) of the point m in terms of the polar tangential coördinates (p, φ). Thus

$$x = p \sin \varphi + p' \cos \varphi,$$

$$y = - p \cos \varphi + p' \sin \varphi.$$

The equation of OM is

$$x \cos \varphi + y \sin \varphi = 0.$$

Therefore the rectangular coördinates (X, Y) of the point M are

$$X = p \sin \varphi, \qquad Y = - p \cos \varphi.$$

Hence the tangent to the pedal of the required curve, i.e., the locus of M, at M has the direction given by

$$\frac{dY}{dX} = - \frac{d(p \cos \varphi)}{d(p \sin \varphi)} = - \frac{p' \cos \varphi - p \sin \varphi}{p' \sin \varphi + p \cos \varphi}, = \tan \Phi \text{ say}.$$

Now by the relation

$$x \cos \varphi + y \sin \varphi = p',$$

mM is equal to p', so that

$$\tan \mu = p/p'.$$

Hence the angular coefficient of mm_1 is given by

$$\tan (\varphi + \mu) = \frac{p' \sin \varphi + p \cos \varphi}{p' \cos \varphi - p \sin \varphi}.$$

Therefore

$$\frac{\pi}{2} + \Phi = \varphi + \mu,$$

i.e., mm_1 and TM make right angles.

Similarly m_1m and T_1M_1, tangent to the pedal at M_1, make right angles. Therefore TM and T_1M_1 are parallel, and the perpendiculars OT and OT_1 dropped from O on TM and T_1M_1 respectively lie on one and the same straight line. Denote the length of OT by P, so that the polar tangential coördinates of the point M are (P, Φ), while its rectangular coördinates are (X, Y).

The equation to mm_1, regarded as passing through m, is

$$\eta + p \cos \varphi - p' \sin \varphi = \tan (\varphi + \mu) \cdot (\xi - p \sin \varphi - p' \cos \varphi),$$

ξ, η being current coördinates. But

$$- \tan(\varphi + \mu) \cdot (p \sin\varphi + p' \cos\varphi) - p\cos\varphi + p'\sin\varphi$$

$$= -\frac{2pp'}{p'\cos\varphi - p\sin\varphi}$$

$$= -2pp'(p^2 + p'^2)^{-\frac{1}{2}}(\cos\mu\cos\varphi - \sin\mu\sin\varphi)^{-1}$$

$$= -2pp'(p^2 + p'^2)^{-\frac{1}{2}}\{\cos(\varphi + \mu)\}^{-1}$$

$$= 2pp'(p^2 + p'^2)^{-\frac{1}{2}}(\sin\Phi)^{-1}.$$

Similarly, from the equation to mm_1, regarded as passing through m_1, we have

$$- \tan(\varphi_1 + \mu_1) \cdot (p_1 \sin\varphi_1 + p_1' \cos\varphi_1) - p_1\cos\varphi_1 + p_1'\sin\varphi_1$$

$$= 2p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}}(\sin\Phi_1)^{-1}.$$

These two expressions must be equal, since they come from the equations to the same straight line mm_1. Hence

$$pp'(p^2 + p'^2)^{-\frac{1}{2}}(\sin\Phi)^{-1} = p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}}(\sin\Phi_1)^{-1}.$$

But

$$\Phi_1 = \pi + \Phi.$$

Therefore

$$pp'(p^2 + p'^2)^{-\frac{1}{2}} + p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}} = 0.$$

Now by a well-known theorem, OM is the mean proportional between OT and OM.* Hence

$$P = p^2(p^2 + p'^2)^{-\frac{1}{2}}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{p^3p' + 2pp'^3 - p^2p'p''}{(p^2 + p'^2)^{3/2}} \cdot \frac{d\varphi}{d\Phi}.$$

But from the relation

$$\varphi + \mu = \varphi + \tan^{-1}\frac{p}{p'} = \frac{\pi}{2} + \Phi,$$

we have

$$\frac{d\varphi}{d\Phi} = \frac{p^2 + p'^2}{p^2 + 2p'^2 - pp''}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{pp'}{(p^2 + p'^2)^{1/2}}.$$

Similarly

$$\frac{dP_1}{d\Phi_1} = \frac{p_1p_1'}{(p_1^2 + p_1'^2)^{1/2}}.$$

* See, e.g., Williamson's Differential Calculus, 1892, p. 228.

But the sum of the right-hand members is equal to zero as has been shown above. Therefore

$$\frac{dP}{d\Phi} + \frac{dP_1}{d\Phi_1} = 0,$$

i.e.,

$$P'(\Phi) + P'(\Phi + \pi) = 0.$$

Integrating with respect to Φ,

$$P(\Phi) + P(\Phi + \pi) = \text{const.}$$

Hence *the pedal of the required curve is a curve of constant breadth.*

The converse can be similarly treated. *Singular solutions* are got by putting

$$\tan(\varphi + \mu) = 0 \qquad \text{or} \qquad = \infty,$$

i.e.,

$$p' \sin\varphi + p\cos\varphi = 0, \qquad \text{or} \qquad p'\cos\varphi - p\sin\varphi = 0,$$

i.e.,

$$p = \frac{\text{const.}}{\sin\varphi}, \qquad \text{or} \qquad p = \frac{\text{const.}}{\cos\varphi}.$$

Therefore the oval included by two confocal parabolas having the same axis, but in opposite senses, is the required curve.

SENDAI, JAPAN,
May, 1920.

MINIMAL SURFACES CONTAINING STRAIGHT LINES.*

By James K. Whittemore.

The Enneper-Weierstrass equations of a minimal surface† S are

$$
\begin{aligned}
x &= \frac{1}{2}\int (1 - u^2)F(u)du + \frac{1}{2}\int (1 - v^2)\phi(v)dv, \\
y &= \frac{i}{2}\int (1 + u^2)F(u)du - \frac{i}{2}\int (1 + v^2)\phi(v)dv, \qquad (1) \\
z &= \int uF(u)du + \int v\phi(v)dv.
\end{aligned}
$$

For a real surface S the functions F and ϕ are conjugate; for a real point with a real tangent plane u and v have conjugate values. The direction cosines of the normal are

$$X = \frac{u + v}{uv + 1}, \qquad Y = \frac{i(v - u)}{uv + 1}, \qquad Z = \frac{uv - 1}{uv + 1}.$$

In the following paper we determine the function F so that a real minimal surface S shall contain one or more given straight lines or portions of such lines. In the first section we use Schwarz's formulas for a minimal surface containing a given curve and having at each point of this curve a given tangent plane. In the second section a different and somewhat simpler method is used, giving the result of the first section in a different form and other results. In both these sections the results are applied to derive certain familiar theorems and several facts believed to be new. In the last section we apply the results to double minimal surfaces showing a connection between these surfaces and minimal surfaces containing straight lines.

§ 1. **Schwarz's Equations.** In 1875 H. A. Schwarz gave the equations of a minimal surface containing a given curve and having at each point of this curve a given tangent plane.‡ If the equations of the given curve are

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

* Presented at the meeting of the American Mathematical Society, New York, April 26, 1919.

† Darboux, Leçons sur la théorié générale des surfaces, vol. 1, 2d ed., p. 344. Eisenhart, *Differential Geometry*, p. 256.

‡ Darboux, l.c., p. 449. Eisenhart, l.c., p. 266.

and the direction cosines of the normal to the surface are $X(t)$, $Y(t)$, $Z(t)$, the equations of the surface may be written

$$\bar{x} = \frac{x(u_1) + x(v_1)}{2} + \frac{i}{2}\int_{v_1}^{u_1}\left[Z(t)\frac{dy}{dt} - Y(t)\frac{dz}{dt}\right]dt, \tag{2}$$

with similar equations for $\bar{y}$ and $\bar{z}$.

We choose as the given curve a straight line parallel to the z axis, writing

$$x = a, \qquad y = b, \qquad z = t; \qquad X = \cos\varphi, \qquad Y = \sin\varphi, \qquad Z = 0,$$

where a and b are constants and φ is a real function of t. We seek to determine $F(u)$ in (1) so that the surfaces given by (1) and (2) shall be identical. Since u, v and u_1, v_1 are the parameters of the minimal curves of the surfaces (1) and (2) respectively we may assume without restriction that u is a function of u_1 alone if the surfaces are the same. A necessary condition for this identity is obtained by equating the partial derivatives with respect to u_1 of corresponding coördinates in (1) and (2). We find

$$\frac{\partial x}{\partial u_1} = -\frac{i}{2}\sin\varphi = \frac{1}{2}(1 - u^2)F(u)\frac{du}{du_1},$$

$$\frac{\partial y}{\partial u_1} = \frac{i}{2}\cos\varphi = \frac{i}{2}(1 + u^2)\cdot F(u)\frac{du}{du_1},$$

$$\frac{\partial z}{\partial u_1} = \frac{1}{2} = uF(u)\frac{du}{du_1},$$

where t is replaced by u_1 in φ. The first two equations give

$$\tan\varphi = i\,\frac{1 - u^2}{1 + u^2},$$

from which $u = e^{\varphi i}$. From the third equation we then have

$$F(u) = \frac{1}{2iu^2\varphi'}.$$

Now since φ is a real function of u_1, the same is true of φ', and the latter is a real function of $\varphi = -i\log u$, so that necessarily

$$F(u) = \frac{i}{u^2}R(i\log u), \tag{3}$$

where R is a real function of its argument. Substituting this value of F in (1) and at the same time

$$\phi(v) = -\frac{i}{v^2}R(-i\log v)$$

it appears that for any real function R we have, for $uv - 1 = 0$, $dx = dy = Z = 0$, so that (3) is both necessary and sufficient that the real surface S, given by (1), contain a portion of a line parallel to the z axis.

We make two applications of the result obtained. First, let us determine what real minimal surfaces applicable to surfaces of revolution or to spiral surfaces contain a portion of a line parallel to the z axis. All such surfaces are given* by taking in (1)

$$F(u) = cu^{-2+m+ni},$$

where c, m, n are constants of which the last two are real; if $n = 0$, S is applicable to a surface of revolution; if $m = 0$, S is a spiral surface. In order that a surface of this kind contain a portion of a line parallel to the z axis

$$iR(i \log u) = cu^{m+ni} = ce^{(m+ni)\log u},$$

from which it follows that $m = 0$ and c is pure imaginary. All such surfaces are then given by $F(u) = iu^{-2+ni}$, and homothetic surfaces. If $n = 0$, S is the right helicoid; if $n \neq 0$, S is a special spiral surface, which has previously been proved to contain part of the z axis.†

As a second application let us determine all real double minimal surfaces containing a portion of a line parallel to the z axis. Real double minimal surfaces are given by (1) when‡

$$F(u) = -\frac{1}{u^4}\phi\left(-\frac{1}{u}\right).$$

We have

$$-\frac{1}{u^4}\phi\left(-\frac{1}{u}\right) = \frac{i}{u^4}u^2 R\left[-i\log\left(-\frac{1}{u}\right)\right] = \frac{i}{u^2}R(i\log u + \pi).$$

It is to be noted that the expression last written has the same form as $F(u)$, which is necessarily the case since it must again give S if used in (1) in place of $F(u)$. For a double surface this expression is equal to $F(u)$; a necessary and sufficient condition is that R have the period π. If we choose (*a*) R constant, S is the right helicoid; (*b*) $R(x) = 2\sin 2x$, $F(u) = 1/u^4 - 1$, S is Henneberg's surface;§ $R(x) = 2\cos 2x$ gives the same surface rotated about the z axis through 45°; (*c*) $R(x) = -\frac{1}{2}\csc 2x$, $F(u) = 1/(1 - u^4)$, S is Scherk's surface;‡ (*d*) $R(x) = -4\sin mx \cos nx$ gives

$$F(u) = \frac{(u^m - u^{-m})(u^n + u^{-n})}{u^2};$$

* Darboux, l.c., pp. 359, 368, 396.

† Transactions, vol. 19, no. 4 (1918), p. 325.

‡ Darboux, l.c., p. 410. See also p. 346.

§ Eisenhart, l.c., pp. 267, 260. Darboux, l.c., p. 327.

S is a double surface if m and n are integers whose sum is even, and is algebraic if m and n are numerically different integers.

§ 2. **Lines Parallel to the xy Plane.** We consider the condition that the real minimal surface (1) contain a line, or a portion of a line, parallel to the xy plane, given by

$$x - y \tan \beta = c_1, \qquad z = c_2,$$

using a method quite different from that of the preceding section. Along this line, supposed to lie on (1),

$$dx - \tan \beta dy = dz = \Sigma X dx = 0.$$

From these equations and (1) it follows

$$(u + v) \tan \beta + i(v - u) = 0, \qquad v = ue^{2\beta i}.$$

Substituting the value of v last given in $dz = 0$ from (1),

$$e^{-2\beta i} F(u) + e^{2\beta i} \phi(ue^{2\beta i}) = 0.$$

In the last equation we write $u = u'e^{-\beta i}$, which then becomes

$$e^{-2\beta i} F(u'e^{-\beta i}) + e^{2\beta i} \phi(u'e^{\beta i}) = 0.$$

Since the two terms of this equation are conjugate for real values of u' we have necessarily

$$e^{-2\beta i} F(u'e^{-\beta i}) = iR(u'),$$

where R is a real function. We may write $F(u)$ in the form

$$F(u) = \frac{i}{u^2} R(ue^{\beta i}).$$

It is easily proved that if $F(u)$ has the form last given the line lies in the surface (1), so that the condition for $F(u)$ is sufficient as well as necessary. In particular if the surface contains a line, or part of a line, parallel to the y axis, $\beta = 0$, $F(u) = iR(u)$; to contain a line, or part of a line, parallel to the x axis, $\beta = \pi/2$, $F(u) = iR(ui)$. The last condition may conveniently be written $F(u) = p(u) + iq(u)$, where p and q are both real functions, p being an odd function and q an even function of u. It is evident that $F(u) = iq(u)$, where q is a real even function of u is both necessary and sufficient that (1) contain lines, or parts of lines, parallel to the x and y axes. The function $F(u)$ has the form last given for the right helicoid, $q = 1/u^2$, evidently the only real minimal surface n-taining lines parallel to all the lines of a plane; for Enneper's surface,* q constant; for Henneberg's surface, $q = 1 + 1/u^4$; for Scherk's surface,

* Darboux, l.c., p. 373. Eisenhart, l.c., p. 269.

$q = 1/(1 + u^4)$. The last three as given are rotated 45° about the z axis from the positions in which they are usually given.

§ 3. **Double Surfaces.** It has been stated in § 1 that the condition that equations (1) give a real double minimal surface is

$$F(u) = -\frac{1}{u^4}\phi\left(-\frac{1}{u}\right),$$

where F and ϕ are conjugate functions. Darboux has given the solution of this equation* in the form

$$F(u) = \frac{1}{u^2}\left[\varphi(u) - \varphi'\left(-\frac{1}{u}\right)\right],$$

where φ and φ' are any two conjugate functions. We have solved the equation in a different form, which may be shown equivalent to Darboux's, as follows:

Let $F(u) = P(u)/u^2$ and $\phi(u) = Q(u)/u^2$, where P and Q are conjugate. The condition for a double surface becomes

$$P(u) = -Q\left(-\frac{1}{u}\right).$$

We write $P = p + iq$ and $Q = p - iq$, where p and q are two real functions of u. The condition is replaced by the two equations,

$$p(u) = -p\left(-\frac{1}{u}\right), \qquad q(u) = q\left(-\frac{1}{u}\right).$$

Replacing u by $(u + 1/u)/2 + (u - 1/u)/2$ we may regard p and q as two real functions of the two variables, $u + 1/u$ and $u - 1/u$. The conditions for a double surface become

$$p\left(u + \frac{1}{u}, u - \frac{1}{u}\right) = -p\left(-\left[u + \frac{1}{u}\right], u - \frac{1}{u}\right),$$

$$q\left(u + \frac{1}{u}, u - \frac{1}{u}\right) = q\left(-\left[u + \frac{1}{u}\right], u - \frac{1}{u}\right).$$

Evidently it is both necessary and sufficient for a real double surface that F and ϕ be conjugate and that

$$F = \frac{1}{u^2}p\left(u + \frac{1}{u}, u - \frac{1}{u}\right) + \frac{i}{u^2}q\left(u + \frac{1}{u}, u - \frac{1}{u}\right),$$

where p is an odd function and q an even function of the argument $u + 1/u$. Henneberg's surface, for example, is given in two different

* L.c., pp. 411, 412.

positions by

$$F(u) = 1 - \frac{1}{u^4} = \frac{1}{u^2}\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right),$$

$$F(u) = i\left(1 + \frac{1}{u^4}\right) = \frac{i}{u^2}\left[\left(u + \frac{1}{u}\right)^2 - 2\right] = \frac{i}{u^2}\left[\left(u - \frac{1}{u}\right)^2 + 2\right].$$

It is clear that in the first expression p is an odd function, in the second both values of q are even functions of $u + 1/u$. It may be noted that the generalization of Henneberg's surface given by Darboux,*

$$F(u) = \frac{1}{u^2}\left(u + \frac{1}{u}\right)^\beta \left(u - \frac{1}{u}\right)^\alpha,$$

β being an odd integer, is an immediate consequence of the preceding result. The general solution of the functional equation for real double surfaces can be given a third form, which may be convenient for some purposes, namely

$$F(u) = \frac{i}{u^2} R\left[u - \frac{1}{u},\, i\left(u + \frac{1}{u}\right)\right],$$

where R is any real function of the two arguments.

We consider again the problem of § 1, to determine $F(u)$ so that the surface given by (1), supposed real, shall contain a line, or part of a line, parallel to the z axis. We use the method of § 2. We require that for $uv - 1 = 0$, $dx = dy = 0$. From $dx = 0$, we find, substituting $v = 1/u$,

$$F(u) = -\frac{1}{u^4}\phi\left(\frac{1}{u}\right),$$

and observe that if this condition is satisfied we have $dy = 0$ along the same line. This equation resembles so closely that for a double surface that it is natural to solve it in the same way. The solution may be given the three forms,

$$F(u) = \frac{1}{u^2}\left[\varphi(u) - \varphi'\left(\frac{1}{u}\right)\right], \qquad \varphi \text{ and } \varphi' \text{ conjugate;}$$

$$F(u) = \frac{1}{u^2}p\left(u + \frac{1}{u},\, u - \frac{1}{u}\right) + \frac{i}{u^2}q\left(u + \frac{1}{u},\, u - \frac{1}{u}\right),$$

p and q real functions of their arguments and odd and even functions respectively of $u - 1/u$;

$$F(u) = \frac{i}{u^2}R\left[u + \frac{1}{u},\, i\left(u - \frac{1}{u}\right)\right],$$

R any real function of its arguments. It may be proved that these

* L.c., p. 421.

results are in agreement with those of § 1. It may also be proved that all of the special forms of $F(u)$ determined in this and the preceding sections are unchanged when $F(u)$ is replaced by $-1/u^4\phi(-1/u)$.

We now prove several theorems concerning straight lines of double surfaces. Suppose first a real double minimal surface contains lines, or parts of lines, parallel to the x and y axes; then

$$F(u) = \frac{i}{u^2} q\left(u + \frac{1}{u},\ u - \frac{1}{u}\right),$$

where q is a real function of both arguments, an even function of $u + 1/u$, and an even function of u. It follows that q is an even function of $u - 1/u$, and the surface must contain a line, or part of a line, parallel to the z axis. We may state the theorem: the necessary and sufficient condition that a real minimal surface, which contains two perpendicular lines, or parts of such lines, be a double surface is that it contain also a line, or part of a line, perpendicular to the two other lines. Real double minimal surfaces containing lines, or parts of lines, parallel to the three coördinate axes are the right helicoid, $F(u) = i/u^2$; Henneberg's surface, $F(u) = i(1 + 1/u^4)$; Scherk's surface, $F(u) = i/(1 + u^4)$.

We prove the following generalization of the preceding theorem: the necessary and sufficient condition that a real minimal surface, which contains two plane geodesics, not straight lines, in perpendicular planes, be a double surface is that it contain a line, or part of a line, parallel to the intersection of the planes of the geodesics. We remark that a plane geodesic of a surface, not a straight line, is necessarily a line of curvature, and that the normals to the surface along such a curve are perpendicular to the normal to the plane of the geodesic. Suppose a real minimal surface contains two such plane geodesics in planes parallel to the xz and the yz planes; the normals to the surface along these curves are perpendicular to the y and x axes respectively, and the equations of these two lines of curvature are then $v = u$, and $v = -u$. On substituting these values in the differential equation of the lines of curvature* of (1),

$$F(u)du^2 - \phi(v)dv^2 = 0,$$

it is easily seen that F must be a real even function of u,

$$F(u) = \frac{1}{u^2} p\left(u + \frac{1}{u},\ u - \frac{1}{u}\right).$$

If the surface is a double surface p is an odd function of $u + 1/u$, consequently also an odd function of $u - 1/u$, and the surface therefore contains a line, or part of a line, parallel to the z axis. Conversely, if p is an even

* Eisenhart, l.c., p. 257.

function of u, and an odd function of $u - 1/u$, it is also an odd function of $u + 1/u$, and (1) is a double surface. Henneberg's and Scherk's surfaces are examples of real minimal double surfaces containing two plane geodesics, not straight lines, in planes parallel to the xz and yz planes, and part of a line parallel to the z axis, when given in the forms,

$$F(u) = 1 - \frac{1}{u^4} = \frac{1}{u^2}\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right),$$

$$F(u) = \frac{1}{1 - u^4} = -\frac{1}{u^2}\,\frac{1}{\left(u + \frac{1}{u}\right)\left(u - \frac{1}{u}\right)}.$$

No real minimal surface contains a line, or part of a line, and a plane geodesic, not a straight line, in a plane perpendicular to this line. The condition that the surface (1) contain a line parallel to the y axis is $F(u) = iR(u)$; that $v = u$ be a line of curvature is $F(u) = R_1(u)$, where R and R_1 are both real functions. These conditions are evidently incompatible.

We prove finally two theorems relating to associate surfaces of a real double minimal surface. A surface associate to the real minimal surface (1) is given by the same equations when $F(u)$ and $\phi(v)$ are replaced by $e^{\alpha i}F(u)$ and $e^{-\alpha i}\,\phi(v)$ respectively, where α is a real constant.* In particular $\alpha = \pi$ gives a surface symmetrical to (1) with respect to the origin, $\alpha = \pi/2$ gives the adjoint surface. We prove first that no surface associate to a real double minimal surface, except the symmetrical surface, $\alpha = \pi$, is a double surface; second, that no surface associate to a real double minimal surface, except the surface itself, $\alpha = 0$, the symmetrical surface, $\alpha = \pi$, and the surfaces adjoint to these, $\alpha = \pm\pi/2$, can contain a straight line or part of a line. As previously stated the surface S_α associate to (1) is given by replacing $F(u)$ in those equations by

$$e^{\alpha i}F(u) = \frac{1}{u^2}[p \cos\alpha - q \sin\alpha + i(p \sin\alpha + q \cos\alpha)],$$

where

$$u^2F(u) = p(h, k) + iq(h, k), \qquad h = u + \frac{1}{u}, \qquad k = u - \frac{1}{u}.$$

If (1) is a double surface p and q are odd and even functions of h respectively. If S_α is also a double surface

$$p \sin\alpha = q \sin\alpha = 0, \qquad \sin\alpha = 0,$$

and $\alpha = 0$ or π.

* Darboux, l.c., p. 379. Eisenhart, l.c., p. 263.

If S_a contains a straight line, or part of a line, we may without restriction suppose the line to be parallel to the z axis. Then $p \cos \alpha - q \sin \alpha$ and $p \sin \alpha + q \cos \alpha$ are odd and even functions of k respectively. We may put this property in evidence by writing

$$p \cos \alpha - q \sin \alpha = kf(h, k^2), \qquad p \sin \alpha + q \cos \alpha = \varphi(h, k^2).$$

Remembering that p and q are odd and even functions of h respectively when (1) is a double surface, and changing the sign of h in the preceding equations, we have

$$-p \cos \alpha - q \sin \alpha = kf(-h, k^2), \qquad -p \sin \alpha + q \cos \alpha = \varphi(-h, k^2).$$

If the first two equations are solved for p and q and the values obtained substituted in the last two the resulting equations may be written

$$kf(-h, k^2) + kf(h, k^2) \cos 2\alpha = -\varphi(h, k^2) \sin 2\alpha,$$

$$\varphi(-h, k^2) - \varphi(h, k^2) \cos 2\alpha = -kf(h, k^2) \sin 2\alpha.$$

Since an even function of k and an odd function of k cannot be equal unless both vanish, we must have $\sin 2\alpha = 0$, and $\alpha = 0, \pm \pi/2$, or π. In order that the real minimal surface (1) be a double surface and contain a line, or part of a line, parallel to the z axis it is both necessary and sufficient that p be an odd function of each of its arguments, h and k, and that q be an even function of each of its arguments. If these conditions are satisfied the symmetrical surface S_π has the same properties. In order that (1) be a double surface and that the adjoint surface $S_{\pi/2}$ contain a line, or part of a line, parallel to the z axis it is both necessary and sufficient that p be an odd function of h and an even function of k and that q be an even function of h and an odd function of k. For example the surface given by (1), when

$$F(u) = \frac{1}{u^2}\left(u + \frac{1}{u}\right),$$

is a real double surface, ϕ being of course the conjugate function, whose adjoint surface contains part of a line parallel to the z axis.

YALE UNIVERSITY,
April, 1920.

AN EXTENSION OF GREEN'S LEMMA TO THE CASE OF A RECTIFIABLE BOUNDARY.*

By Edward B. Van Vleck.

The current proofs of the fundamental Green's Lemma,

$$(1) \qquad \int\int_K \frac{\partial P(x, y)}{\partial x} dy dx = \int_C P(x, y) dy,$$

introduce notable restrictions upon the boundary C of K, the usual restriction being the hypothesis that the boundary is cut by a parallel to the axis of x in only a finite number of points. So far as I have ascertained,† no proof has been previously given covering the general case in which the boundary is merely restricted by the requirement that it shall be rectifiable. Such a proof is given in this note, and furthermore, no condition is imposed upon $P(x, y)$ except its continuity and the integrability of its derivative $\partial P/\partial x$ over the two-dimensional field K.

Cauchy's basal theorem for an analytic function,

$$\int_C f(z) dz = 0,$$

then follows in the usual way, being thus extended to the case of a rectifiable boundary. It should, however, be remarked that this theorem is also established in Jordan's Cours d'analyse‡ under the hypothesis of a rectifiable boundary but independently of Green's Lemma.

For simplicity of presentation the proof of Green's Lemma is first given for a simply connected region bounded by a *simple* closed rectifiable curve, that is, one without double points. The unessential restriction of a boundary without double points and of a region with simple connection is then quickly removed. The method of proof adopted is capable of extension to corresponding theorems in three or more dimensions.

Consider a region K of the plane bounded by a *simple* closed rectifiable curve C:

$$x = \phi(t), \qquad y = \psi(t) \qquad (t_0 \leqq t \leqq t'; \ \phi(t_0) = \phi(t'), \ \psi(t_0) = \psi(t')).$$

* Presented to the American Mathematical Society March 29, 1919.

† For extensions of a different nature see Cino Poli, Atti della r. Academia delle Scienze di Torino, vol. 49 (1913–4), pp. 248–258, and M. Picone, Rendiconti del Circolo Matematico di Palermo, vol. 43 (1918–9), p. 239.

‡ Vol. 1, second edition, pp. 185–191.

For convenience we suppose that a point O within the curve is taken as the origin of coördinates, also that the curve is described in a positive direction when t passes from t_0 to t'. Inclose C in a rectangle with sides parallel to the axes. Then subdivide the rectangle by means of two systems of parallels,

$$x = m/2^n, \qquad y = m'/2^n \qquad (m, m' = 0, \pm 1, \pm 2, \cdots).$$

Thereby it is divided into squares of side $1/2^n$. If n is a sufficiently large integer, O will belong to four initial squares, all of whose points *inclusive of their boundary points* are interior to C. To these initial squares we will annex any other square which borders one of the initial squares along a side and which is entirely free from points of C. If one of the annexed squares is similarly bordered by a square entirely free from points of C, we will annex this square and continue the annexation until it is no longer possible to annex squares entirely free from points of C. In this manner we obtain a connected region consisting only of points interior to C, which we will call the *checkerboard net* K_n belonging to the origin O for a given value of n. Obviously K_n is included in K_{n+1}.

This checkerboard net is a *simply connected* region bounded by a *simple* polygon whose sides are parallel to the axes. To see this, trace any boundary line of the checkerboard net until it first returns upon itself at some point P. The portion included between two successive passages through P is a simple closed polygon whose sides are parallel to the axes and consist of segments of length $1/2^n$. Clearly the interior of this polygon, like the perimeter itself, is interior to C. Now each segment of the perimeter separates a square S' of the checkerboard net containing only points interior to C from a square S'' which contains at least one point of C. The squares inside the polygon therefore belong to the checkerboard net. Since also no square exterior to the polygon can be connected with one in the interior by a chain of contiguous squares without including an S'', this polygon is the whole of the checkerboard net K_n. The latter is accordingly a simply connected region bounded by a simple polygon, as stated.

It can next be shown that when n is sufficiently increased, K_n may be made to include any assigned point P interior to C. For let P be connected with O by some continuous curve OP lying wholly within C. Denote with δ the minimum distance between points of OP and points of C, and increase n so that the length of the diagonals of the squares composing K_n shall be smaller than δ. Then any point of OP will either lie within a square containing no points of C or lie on a common side of two such squares or be the vertex common to four such squares. The line OP must therefore lie within K_n.

Since C is rectifiable, it is also squarable; in other words, the total area of our squares containing points of C can be made smaller than a prescribed ϵ by sufficiently increasing n. Denote with A the area within C. The total area of all squares wholly interior to C will be greater than $A - \epsilon$. For fixed $n = N$ this total area may consist of a number of separate pieces which are composed of squares of side $1/2^N$. If a point P is selected in any one of these component squares, it will be included within K_n when n is sufficiently increased, and simultaneously with P the whole of the component square. Hence all the squares composing our fixed area will ultimately be included in K_n. Consequently by sufficiently increasing n the area of K_n may be made to differ from that within C by less than ϵ.

Consider now a function $P(x, y)$ which is continuous over the closed field K bounded by C and has a derivative $\partial P(x, y)/\partial x$ which is properly integrable over this field. By a well-known theorem* the existence of the proper double integral $\int\int(\partial P/\partial x)dxdy$ over a rectangular field with sides parallel to the axes carries with it the existence of the two iterated integrals and their equality to the double integral. Now this iterated integral $\int dy \int(\partial P/\partial x)dx$ is equal to the curvilinear integral $\int P(x, y)dy$ taken in a positive direction around the boundary, the components of this curvilinear integral over the horizontal sides of the rectangle being, of course, equal to zero. Hence Green's Lemma (1) holds for each square of our checkerboard net. By addition of the squares we obtain

$$(2) \qquad \int\int_{K_n} \frac{\partial P}{\partial x}\, dxdy = \int_{C_n} P(x, y)dy,$$

where C_n denotes the boundary of K_n.

The remainder of our work consists of the extension of Green's Lemma from K_n to the field K by passing to the limit with indefinitely increasing n.

By hypothesis $\partial P/\partial x$ is properly integrable over K and has therefore an upper limit M to its absolute value. By sufficiently increasing n the difference of the areas of K and K_n can be made smaller than a prescribed ϵ. Then the left hand member of (2) will differ from that of (1) by less than ϵM. Consequently the left hand member of (2) approaches the left-hand member of (1) as its limit when n is indefinitely increased.

Before making a comparison of the right-hand members of (1) and (2) it may be remarked that the former is given its natural meaning and is accordingly a Stieltjes integral obtained as follows. Let the interval $(t_0, t' = t_{n+1})$ be subdivided by successive values $t_1, t_2, \cdots, t_k, \cdots, t_n$, and denote by $R_k = (x_k, y_k)$ the corresponding points of C. Form the sum

$$(3) \qquad \sum_{k=0}^{n} P(\xi_k, \eta_k)\cdot(y_{k+1} - y_k),$$

* Cf. Hobson, The Theory of Functions of a Real Variable, § 314, p. 425.

in which (ξ_k, η_k) denotes a point taken arbitrarily on C between R_k and R_{k+1} inclusive. If this sum tends to a limit when n increases and the maximum size (norm) of the subintervals $|t_{k+1} - t_k|$ is indefinitely decreased, the limit is the integral denoted by the right hand member of (1). This limit exists when $P(x, y)$ is continuous and $\psi(t)$ is a function of limited variation.* In the case before us the variation

$$\sum_{k=0}^{n} |\psi_{k+1}(t) - \psi_k(t)| = \sum_{k=0}^{n} |y_{k+1} - y_k|$$

is limited because this sum is less than the length L of our rectifiable curve C.

Suppose that the t-norm has been taken so small that the difference between the approximation (3) and the right-hand member of (1) is less in absolute value than an assigned ϵ. *Without infringing this requirement* we may modify the partition $R_0R_1R_2 \cdots R_{n+1}$ of C and correspondingly the sum (3) in the following manner. If any division R_kR_{k+1} of C is a horizontal segment, the corresponding term in (3) is zero since $y_{k+1} = y_k$. Each succession of such horizontal divisions may therefore be merged into a single horizontal division R_kR_{k+1} without changing the value of (3). If then an adjoining piece $R_{k+1}R''$ of the division following (similarly of the division just preceding) is also a horizontal segment, we will interpolate between t_{k+1} and t_{k+2} a division point t'' corresponding to R'', modifying at the same time the sum (3) correspondingly. Then the horizontal piece $R_{k+1}R''$ can be merged with R_kR_{k+1}. In this manner let the horizontal divisions of our partition be made of maximum length. Each horizontal division R_kR_{k+1} is then followed (and similarly preceded) by a division in which there are points (x, y) as near to R_{k+1} as we please for which $|y - y_{k+1}| \neq 0$. Our partition of C is then *normalized*.

We now proceed to partition the boundary of our checkerboard net. Start from any corner P_0 of K_n and trace its boundary C_n in a positive direction. As we pass successively through vertices of component squares of K_n, denote these vertices by $P_1, P_2, P_3, \cdots$, omitting in this enumeration all vertices on each horizontal side of C_n except its extremities. The vertical sides of C_n are thus divided into segments of length $1/2^n$, while the horizontal sides are multiples of this length. (Fig. 1.)

To this partition of C_n we will make correspond a partition $Q_0Q_1Q_2 \cdots$ of C in the following manner. Every line $y = m/2^n$ which crosses or bounds K_n ceases to cross or bound it at a point P_i. We will take as the corresponding point Q_i the first point in which the line meets C after emergence at P_i. (See Fig. 1.) A division point Q_i of C is thus obtained

* Cf. Vallée Poussin, Cours d'analyse infinitésimale, vol. 1, 3d ed., § 346.

for every P_i except those at which the interior of K_n makes a 270° bend. Suppose, if possible, the line to meet C_n between P_i and Q_i, and let P' be the first point of meeting. Then P_iP' and one or the other of the two portions of C_n between P_i and P' would together form a simple closed polygon whose interior is entirely free from points of C and should therefore belong to the checkerboard net, whereas it lies without. It follows that the interior of the horizontal segment P_iQ_i is exterior to K_n. Any

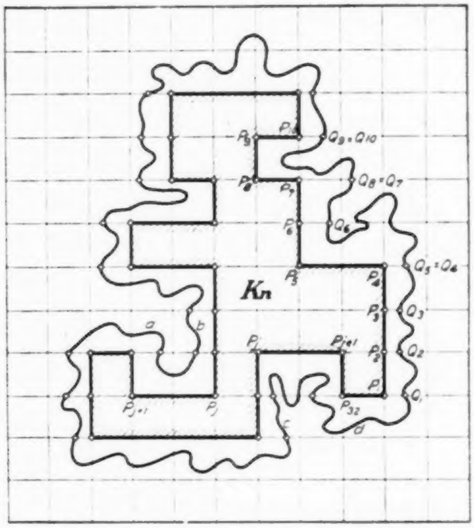

FIG. 1.

two such lines P_iQ_i and $P_{i+k}Q_{i+k}$ with the included part

$$P_iP_{i+1}P_{i+2} \cdots P_{i+k}$$

of C_n and the corresponding portion of C will bound a simply connected region interior to C and exterior to K_n. Into this region the lines $P_{i+j}Q_{i+j}$ $(j < k)$ penetrate. Since the points Q_{i+j} are the points where these lines first meet C, they must be included on C between Q_i and Q_{i+k}. It follows that *the order of the Q_i on C is exactly the same as the order of the corresponding P_i on C_n.*

It remains yet to fix a point Q_i on C corresponding to a vertex P_i of K_n at which the angle of the polygon is 270°. If at the other extremity of the horizontal side which terminates in such a P_i the angle is not 270°, we have already fixed a point Q corresponding to that extremity. We will then make this same point correspond also to P_i so that we have either $Q_{i-1} = Q_i$ or $Q_{i+1} = Q_i$. On the other hand, when the angles at both extremities of a horizontal side P_jP_{j+1} are 270°, we will slightly modify

our polygon K_n in order that the corresponding points Q_j, Q_{j+1} on C shall have the same ordinate as P_j and P_{j+1}. According as the exterior of K_n lies just above or below P_jP_{j+1}, we will move this side parallel to itself upward or downward until it first contains a point of C (see Fig. 2). The amount of this vertical displacement can not exceed $1/2^n$. The alteration in K_n merely increases its area and therefore does not affect the validity of our previous comparison between the double integrals (1) and (2). We will now take as our points Q_j, Q_{j+1} the points* of C

FIG. 2.

on the new and displaced side P_jP_{j+1} which are nearest to P_j and P_{j+1} respectively. Thus Q_j, Q_{j+1} are inserted in order on C between Q_{j-1} and Q_{j+1}.

It will next be shown that the Q_i remain partition points of C when n is increased by 1. The segments P_iQ_i were, in fact, the portions of the lines $y = m/2^n$ which are included between C_n and C. Now these lines are included among the lines $y = m/2^{n+1}$. Since K_{n+1} contains K_n, the segment P_iQ_i is merely replaced by the portion of the segment included between K_{n+1} and C when n is replaced by $n + 1$. Hence the Q_i continue to be partition points when n is increased. To see that this is also true of Q_j and Q_{j+1}, consider the squares exterior to the (unmodified) K_n which border it along P_jP_{j+1} before displacement of this side. Since these are of side $1/2^n$, each is divided into four subsquares when n is increased by 1. Some of these subsquares may be annexed to K_n to form K_{n+1}. If Q_j, Q_{j+1} lie in subsquares adjacent to P_jP_{j+1}, it is impossible to annex all the adjacent subsquares. Hence either the whole of P_jP_{j+1} or portions of the same are included as sides in the boundary of K_{n+1}. The angles of K_{n+1} at the extremities of these sides are obviously $270°$, and Q_j, Q_{j+1} are the points of C nearest to one or two of these sides. Thus Q_j, Q_{j+1} play the same rôle relatively to K_{n+1} as to K_n. On the other hand, if none of the subsquares adjacent to P_jP_{j+1} contain either Q_j or Q_{j+1}, all these subsquares may be annexed to K_n, whereby P_jP_{j+1} is displaced by an amount $1/2^{n+1}$. Then Q_j and Q_{j+1} lie in subsquares adjacent to the displaced P_jP_{j+1} and the situation is exactly the same as that just described. Accordingly Q_j, Q_{j+1} have the same relation to the unmodified K_{n+1} as

* The two points may, of course, coincide. It may be further remarked that when the displacement of the side is exactly $1/2^n$, we have a coincidence of P_j, P_{j+1} with P_{j-1}, P_{j+2} and correspondingly of Q_j, Q_{j+1} with Q_{j-1}, Q_{j+2}.

to the unmodified K_n, and they must therefore be retained as division points of C when n is increased.

Parenthetically it may be remarked that our final checkerboard net K_n, obtained through modification in the manner above described, is included in the modified K_{n+1}.

We will next see that *when n is sufficiently increased, the division points Q_i enter into any portion of C which is not a horizontal segment.* This is true, for example, even if horizontal segments are everywhere dense in the portion considered. For proof, take any two points Q', Q'' with different ordinates in the portion under consideration; and let $m/2^n$ be an intermediate ordinate. Draw the line $y = m/2^n$ and suppose Q' to lie below this line, Q'' above. A sufficiently small vicinity of any point of C below the line can not contain any points interior to K which are above the line. As we pass along C from Q' to Q'', we must come either to a last point of this nature or to a first point $\bar{Q}$ whose vicinity, no matter how small, will contain interior points of K above the line. The former alternative is impossible since the point would be the limit of points with vicinities of both characters, and any vicinity of the point would therefore contain interior points of K both above and below the line. Clearly $\bar{Q}$ lies on the line $y = m/2^n$. Consider now an interval of the line having $\bar{Q}$ as its center and a length δ less than the minimum distance from $\bar{Q}$ to any point of C not included in the portion $Q'Q''$. We will establish the intuitive fact that the interior of K must cut across the interval. Suppose, if possible, that it does not. Describe a circle of radius $\delta/2$ about $\bar{Q}$ as center. The semi-perimeter of this circle below the interval is crossed one or more times by the interior of K since any vicinity of $\bar{Q}$ contains points interior to K and below $y = m/2^n$. Take a piece $A\bar{Q}B$ of the arc $Q'\bar{Q}Q''$ of C with a length less than $\delta/2$, A being supposed to lie between Q' and $\bar{Q}$. This piece will contain all points of C sufficiently near to $\bar{Q}$ since C is by hypothesis without double points. Hence $A\bar{Q}$ will be part of the boundary of one of the pieces of the interior of K which penetrate into the semicircle below our interval. Then $\bar{Q}B$ is the continuation of its boundary. Since by hypothesis the interior of this piece does not cross the interval, it follows that all points of $\bar{Q}B$ lie on or below the line $y = m/2^n$. Now no other portion of K exterior to this piece can come within a sufficiently small vicinity of $\bar{Q}$ since C is without double points. Consequently in a sufficiently small vicinity of $\bar{Q}$ there can not lie any points of K above $y = m/2^n$, which gives a contradiction. We conclude therefore that the interior of K crosses the δ-interval, as stated, and cuts out one or more subintervals.

Consider any one of these subintervals. Let J be an interior point. When n is sufficiently increased, J will be included within our checker-

board net K_n. The vector $J\overline{Q}$ emerges then from K_n between J and $\overline{Q}$, and the first point in which it meets C after emergence will be a division point Q_i of C. This point lies in $Q'QQ''$. Hence the points Q_i will enter into any portion of C which is not a horizontal segment when n is sufficiently increased.

Our partitions $P_0P_1P_2 \cdots$ and $Q_0Q_1Q_2 \cdots$ of C_n respectively were so formed that corresponding points P_i and Q_i have the same ordinate y_i. With respect to these partitions we will now form approximating sums for $\int_{C_n} P(x, y)dy$ and $\int_C P(x, y)dy$. To prove then that the latter integral is the limit of the former when n is increased indefinitely, it will suffice to show that the difference between these two approximations, and between each approximation and the corresponding integral, can be made arbitrarily small by sufficiently increasing n.

Construct first with respect to C_n the sum

$$\sum_i P(\xi_i', \eta_i')\cdot(y_{i+1} - y_i), \tag{4}$$

in which y_{i+1}, y_i are the ordinates of P_{i+1}, P_i and (ξ_i', η_i') is an arbitrarily chosen point of P_iP_{i+1}. In this sum all terms vanish except those which relate to vertical segments P_iP_{i+1} of C_n. These are of length $1/2^n$. Now the quadrilateral bounded by the linear segments P_iP_{i+1}, P_iQ_i, $P_{i+1}Q_{i+1}$ and the arc Q_iQ_{i+1} of C is exterior to K_n and contains in its interior no points of C. Also in the checkerboard division of K each vertical segment P_iP_{i+1} of K_n borders a square exterior to K_n which contains at least one point of C. This can only be if the arc Q_iQ_{i+1} enters or touches the square and accordingly contains at least one point (ξ_i, η_i) in the square. Take such a point (ξ_i, η_i) for every arc Q_iQ_{i+1} corresponding to a vertical P_iP_{i+1} and an arbitrary point of the arc when $y_i = y_{i+1}$, and construct the sum

$$\sum_i P(\xi_i, \eta_i)\cdot(y_{i+1} - y_i). \tag{5}$$

Since $P(x, y)$ is supposed continuous over K, the variation of $P(x, y)$ can be made less than an arbitrarily assigned ϵ simultaneously in every checkerboard divisions of K by making the norm $1/2^n$ sufficiently small, i.e., by making n sufficiently large. Then the difference of the approximations (4) and (5) is numerically less than $\epsilon\Sigma_i|y_{i+1} - y_i|$. Since $|y_{i+1}-y_i|$ is the vertical distance between two consecutive divisions points Q_i, Q_{i+1} of C, this quantity is less than ϵL, where L denotes the length of C. Furthermore, by the same uniform continuity each term

$$P(\xi_i', \eta_i')\cdot(y_{i+1} - y_i)$$

in (4) will differ from the value of $\int_{y_i}^{y_{i+1}} P(x, y)dy$, taken along C_n between P_i and P_{i+1}, by less than $\epsilon\, |y_{i+1} - y_i|$. Consequently the values of (4) and of $\int_{C_n} P(x, y)dy$ can also be made to differ by less than the arbitrary quantity ϵL by sufficiently increasing n.

It remains now only to prove that the sum (5) can be made to differ from $\int_C P(x, y)dy$ by as little as we wish. To establish this, let us compare (5) with an approximation (3) for the latter which corresponds to a partition $R_0R_1R_2 \cdots$ of C normalized in the manner previously described. We will suppose that the t-norm for the partition is taken so small that the variation of $P(x, y)$ in each R_kR_{k+1} (with the exception of horizontal divisions) is less than an assigned ϵ'', and also so small that the approximation (3) differs from $\int_C P(x, y)dy$ as little as we wish. Denote by ν the number of horizontal divisions R_kR_{k+1} which are horizontal segments. If now, as we shall suppose, n has been taken sufficiently large, the Q_iQ_{i+1} will be found in every R_kR_{k+1} except the ν horizontal divisions, and, furthermore, the distances along C from the terminal points R_k, R_{k+1} of the horizontal divisions to the first Q_i in the divisions respectively preceding and following may be supposed less than an arbitrarily assigned ϵ'. To facilitate comparison between (3) and (5), put in (3)

$$y_{k+1} - y_k = (y_i - y_k) + (y_{i+1} - y_i) + \cdots + (y_{k+1} - y_{i+j}),$$

when points $Q_i, Q_{i+1}, \cdots, Q_{i+j}$ are contained in R_kR_{k+1}. Similarly write

$$y_{i+1} - y_i = (y_k - y_i) + (y_{i+1} - y_k),$$

when Q_i, and Q_{i+1} are separated on C by a R_k, and

$$y_{i+1} - y_i = (y_k - y_i) + (y_{k+1} - y_k) + (y_{i+1} - y_{k+1}),$$

when separated by a horizontal segment R_kR_{k+1}. We have thus split up (3) and (5) for comparison into an equal number of terms corresponding to the same subdivisions of C.

These subdivisions are of four sorts. First, there are the ν horizontal divisions of the R-partition. The corresponding terms in (3) and (5) vanish since the ordinates of the extremities of such a division are equal. Secondly, there are 2ν adjoining subdivisions, each less than ϵ' in length. If M denotes the maximum of the absolute value of $P(x, y)$ upon C, the sum of the 2ν components in (3) or in (5) corresponding to these subdivisions will not exceed $2\nu\epsilon' M$ in absolute value. Thirdly, we have as

subdivisions such of the Q_iQ_{i+1} as lie each entirely in some R_kR_{k+1}. The corresponding terms in (3) and in (5) will then not differ by as much as $\epsilon'' |y_{i+1} - y_i|$. Lastly, we have subdivisions Q_iR_k and R_kQ_{i+1} due to separation of Q_i and Q_{i+1} by an R_k. Since the multipliers of $y_k - y_i$ and $y_{i+1} - y_k$ in (5) are values of $P(x, y)$ taken at points of Q_iQ_{i+1} and hence taken from the same or adjacent divisions $R_{k-1}R_k$ and R_kR_{k+1}, they differ from the corresponding multipliers in (3) by less than $2\epsilon''$. It follows that the values of (3) and (5) differ by less than

$$4\nu\epsilon' M + \epsilon''\Sigma|y_{i+1} - y_i| + 2\epsilon''\Sigma(|y_k - y_i| + |y_{i+1} - y_k|)$$
$$< 4\nu\epsilon' M + \epsilon''(L + 2L).$$

Now we first prescribed ϵ'' and then *fixed* an R-partition with some value of ν. Then by sufficiently increasing n we made ϵ' and therefore $\epsilon'\nu$ as small as we please. Hence by prescribing ϵ'' and increasing n we may make (3) and (5) to differ as little as we choose. Since also (3) was chosen arbitrarily near to $\int_C P(x, y)dy$, we conclude that when n is indefinitely increased, the sum (5) approaches this integral as its limit. This completes the proof that the limit of the right-hand member of (2) is the right-hand member of (1) and establishes Green's lemma for the case of a simple rectifiable curve.

Extension can now be quickly made to the general case of a simply connected region bounded by a rectifiable curve C. Multiple points of C may exist, and portions of it may be bordered by the interior of K on opposite sides, being twice traced and in opposite directions when C is completely described. The preceding proof applies except for the two paragraphs which show that the Q_i enter with increasing n into any portion of C which is not a horizontal segment. To establish this we may make use of the fact that when the boundary of a simple connected region is the continuous image of a circle, all of its points are "*attainable*."* In other words, any point of the boundary can be reached from any interior point by a polygonal line lying entirely in the interior (except for its end point on the boundary) or approached as limit point by a polygonal line consisting of an infinite number of interior segments. Since C is rectifiable and continuous, it is the continuous image of a circle, and all of its points are consequently attainable from the interior of K.

Consider now any portion of C which is not a horizontal segment and, as before, take on it any two points Q', Q'' with different ordinates. From

* Schoenflies, Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, Zweiter Teil, p. 189; Jahresbericht der deutschen Mathematiker Vereinigung, 1908.

If the correspondence between the boundary and circle is also one-to-one, the former is a simple closed curve.

any point S in the interior of K draw non-intersecting polygonal lines SQ' and SQ'' which "attain" to Q' and Q'' as end- or limit-points. These two lines together, like a cross cut, divide the interior into two separate regions. For if there remains a connected whole region, we could draw in its interior a simple closed polygon starting at S on one side of the lines SQ', SQ'' and terminating at S on the other side. One of these lines, entering the interior of the polygon would have to remain in it, which contradicts the hypothesis that they attain to Q' and Q'' respectively.

Consider now the region bounded by $Q'SQ''$ and the arc $Q'Q''$ of C under consideration. Let $m/2^n$ be any ordinate intermediate between those of Q', Q'' and draw as above the line $y = m/2^n$. The interior of our region cuts out a finite or countably infinite number of intervals on this line. We will show that at least one of these intervals has an end point on the arc $Q'Q''$. Suppose, if possible, the contrary. Then every interval, considered as a crosscut, divides the region into two parts, for one of which the arc $Q'Q''$ is a part of the boundary while for the other it is not. Any interior point of the region whose minimum distance from the arc $Q'Q''$ is less than that to $Q'SQ''$ must obviously lie in the former of the two parts. Draw now in the interior of the region a polygonal path connecting two such points A and B, the former of which lies below $y = m/2^n$ and the other above. This path crosses any one of the above intervals only a finite number of times. Suppose it to cross an infinite number of the intervals. As we are concerned only with a limited portion of the plane, this infinite set of intervals on $y = m/2^n$ must have the lower limit zero to their length, and there must be at least one limit point for the set. A limit point, on the one hand, must lie on $Q'SQ''$, being the limit of end-points of our intervals which themselves lie upon it; on the other hand, since the continuous path approaches to within an infinitesimal distance of the limit point, this point must lie on the path and hence be an interior point of the region. From this contradiction it follows that the path can cross only a finite number of our intervals. Now if in passing from A to B the path crosses any interval, it must ultimately cross back again, for otherwise it would remain in the subregion bounded by the interval and the included piece of $Q'SQ''$ and hence could not reach B. Replace the portion of the path between the first point of crossing an interval and the last point of crossing the same interval by the segment of the interval between these two points. If the modified path still crosses an interval, do the same thing for the first remaining interval it crosses, and so on. We have finally a path which does not cross any one of our intervals and thus connects A and B on opposite sides of the line $y = m/2^n$ without crossing the line. Since this is impossible, our hypothesis is

untenable. We conclude therefore that some one of our intervals has an end point Q on the arc $Q'Q''$ of C.

We may now argue precisely as in the case of a simple rectifiable boundary that this point Q ultimately becomes a division point Q_i of C. If, namely, we taken an interior point $\bar{Q}$ of the interval which ends in Q, then $\bar{Q}$ will become a point of our checkerboard net when n is sufficiently increased. Since Q is the first point in which $y = m/2^n$ meets C after an emergence from the net, we conclude that Q is now a division point Q_i. The remainder of our proof then applies as before.

Extension can be readily made to the case of a multiple connected region bounded by n rectifiable curves, over which $P(x, y)$ is uniquely defined. For this purpose the region can be rendered simply connected by drawing rectilinear cuts between pairs of boundaries. These cuts are without effect upon the right and left hand members of (1).

UNIVERSITY OF WISCONSIN

PERIODIC CONJUGATE NETS.

By Edward S. Hammond.

Introduction. If n functions of u and v, $x^{(1)}$, $x^{(2)}$, $\cdots$, $x^{(n)}$, which satisfy an equation of Laplace of the form

$$(1) \qquad \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial \log a}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial x}{\partial v} + cx,$$

be interpreted as the homogeneous coördinates of a surface in $(n-1)$ space, the parametric curves on this surface are said to form a *conjugate net*. Where no ambiguity arises, this system of curves or the surface on which it lies will be called simply the net N. Equation (1) will be called the *point equation* of N. Now the functions* x_1 and x_{-1}, given by

$$(2) \qquad x_1 = \frac{\partial x}{\partial v} - \frac{\partial \log a}{\partial v}x, \qquad x_{-1} = \frac{\partial x}{\partial u} - \frac{\partial \log b}{\partial u}x,$$

are also homogeneous coördinates of nets, N_1 and N_{-1}, which are called the *first* and *minus first Laplace transforms of* N. N_1 has as its first and minus first Laplace transforms nets N_2 and N itself; N_2 is called the second Laplace transform of N. Developing these transforms in both senses we get a series of nets $\cdots N_{-s}, \cdots, N_{-1}, N, N_1, \cdots, N_r, \cdots$, called a sequence of Laplace.† This sequence will be called the sequence N_r. In the first section of this paper general properties of this sequence will be developed.

In section 2, we impose upon the sequence N_r the condition that it shall be periodic; that is, that a certain Laplace transform N_p of N shall coincide with N itself. After transformation of parameters it is shown that the identity of N_p and N involves the identity of N_{p-1} and N_{-1} and in general, of N_{p-k} and N_{-k}, $k = 0, 1, 2, \cdots, p$. Necessary conditions on the coefficients of the point equation of N are derived and it is shown by discussion of the completely integrable systems of partial differential equations involved that these conditions are also sufficient. It is also shown that if an equation of Laplace of form (1) is the point equation of one periodic net, it is the point equation of an infinity of others of the same period.

The remainder of the paper is taken up with other sequences of Laplace

* Here x_1 indicates any or all of $x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}$. A similar usage is followed throughout.

† Darboux, Leçons sur la théorie générale des surfaces, 2d ed. (1915), vol. II, chap. 2.

closely related to the sequence N_r. The sequences studied in section 3 involve certain properties of families of lines in higher ordered spaces which we proceed to develop. The lines joining corresponding points of a net N and its first Laplace transform N_1 form a two-parameter family G, each line of which is a common tangent of these surfaces. Consider for the sake of definiteness the line joining the points on N and N_1 with parameters u_0 and v_0. Through this line pass two developable surfaces all of whose generators are lines of G, namely, the tangent surfaces of the curve $u = u_0$ on N, and of the curve $v = v_0$ on N_1. When a two-parameter family of lines in higher ordered space possesses either of these equivalent properties, namely, that each line of the family is a common tangent to two surfaces, and that through each line there pass two developable surfaces all of whose rectilinear generators are lines of the family, it is called a *congruence*. In 3-space any two parameter family of lines possesses these properties, but in space of higher order this is not the case. The surfaces to which all the lines of G are tangent are called the *focal surfaces* and the nets N and N_1 the *focal nets* of the congruence.

Levy* has shown that the functions ξ and η, defined by

$$(3) \qquad \xi = x - \frac{\theta}{\frac{\partial \theta}{\partial v}} \frac{\partial x}{\partial v}, \qquad \eta = x - \frac{\theta}{\frac{\partial \theta}{\partial u}} \frac{\partial x}{\partial u},$$

where θ is any solution of (1), may be interpreted as the coördinates of nets which will be called *Levy transforms* of N by means of θ. The points of these nets lie on the lines joining the corresponding points of N and N_1, N_{-1} and N, respectively, and the developables of the congruences so generated cut the surfaces of the nets in the curves of the nets. In section 3, it is shown that these nets, there called $N_{0,1}$ and $N_{-1,0}$, are Laplace transforms of one another. It is also shown that $N_{0,1}$ is a Levy transform of N_1 by means of θ_1, a solution of the point equation of N_1 formed from θ by the same process by which the coördinates of N_1 were formed from those of N. From these properties follows a very intimate connection between the two sequences of Laplace, N_r, the original sequence, and $N_{r,r+1}$ of which $N_{-1,0}$ and $N_{0,1}$ are two nets. The sequence $N_{r,r+1}$ is called the *first Levy sequence*. On it may be formed a first Levy sequence, $N_{r,r+2}$ which is called a *second Levy sequence* of N. The treatment given in section 4 of these sequences and of the Levy sequences of higher orders which are analogously formed, indicates their close dependence upon the Laplace transforms of N. They are actually the sequences of derived nets of higher orders studied by Tzitzeica† and others.

* Levy, Journal de l'École Polytechnique, Vol. LVI (1886), p. 67.

† Tzitzeica, Comptes Rendus, vol. 156 (1913), p. 375.

In section 5, the results of section 2 are applied to these Levy sequences and conditions for their periodicity are derived. Two interesting geometric configurations arising under special conditions are discussed.

As a property of the Levy transforms of a net N, it was mentioned that the developables of the congruences of tangents to the parametric curves of N cut the surfaces of the Levy transforms in the curves of the nets. Whenever this relation holds between a congruence and a net, they are said to be *conjugate*. Two nets conjugate to the same congruence are said to be *in relation* T and the transformation which carries one such net into the other is called a *transformation* T, to use the terminology of Eisenhart* who has developed a general theory of such transformations. The congruence to which both nets are conjugate is called the *conjugate congruence* of the transformation. In section 6, it is shown that similar Laplace transforms of two nets in relation T are also in relation T, and hence that two sequences of Laplace may be developed such that corresponding nets of these sequences are in relation T. The problem of finding a sequence $\bar{N}_r$ so related to the original sequence N_r is reduced to the problem of finding a solution ϕ of the adjoint equation of (1) and quadratures. Owing to arbitrary constants arising in the quadratures, their integration gives a multiple infinity of such sequences between which certain geometric relations exist.

The results of section 2 are then applied to these sequences, and it is found, first, that if equation (1) has periodic solutions, so has its adjoint; second, if such a solution ϕ be used in the determination of $\bar{N}_r$, the sequences $\bar{N}_r$ are also periodic of period p.

1. **Sequences of Laplace.** In the study of these sequences, two functions of the coefficients of equation (1), H and K, defined by

$$
(4) \qquad
\begin{aligned}
H &= -\frac{\partial^2 \log a}{\partial u \partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c,\\
K &= -\frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c,
\end{aligned}
$$

are of constant occurrence. Their most important property is in connection with the transformation to other coördinates x', such that

$$
(5) \qquad x = \lambda x',
$$

where λ is a function of u and v. Since the coördinates x are homogeneous, evidently this transformation has no effect on the net. The coördinates x' do not satisfy equation (1), however, but are solutions of

* Eisenhart, Trans. Amer. Math. Soc., vol. 18 (1917), p. 97.

$$(6)\quad \begin{aligned}\frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial}{\partial v}\log\frac{a}{\lambda}\frac{\partial\theta}{\partial u} + \frac{\partial}{\partial u}\log\frac{b}{\lambda}\frac{\partial\theta}{\partial v}\\ &+\left(-\frac{\partial^2}{\partial u\partial v}\log\lambda - \frac{\partial}{\partial v}\log\frac{a}{\lambda}\frac{\partial}{\partial u}\log\frac{b}{\lambda} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c\right)\theta,\end{aligned}$$

as may be shown by differentiation. If the functions H and K be formed from the coefficients of (6) and the resulting expressions reduced, it is found that they are identical with (4), that is, H and K are invariant under the transformation (5). They are called the *Laplace-Darboux invariants* of the equation (1) or of the net N. If the independent variables are changed by a transformation

$$(7)\quad u = \phi(u'), \qquad v = \psi(v'),$$

the invariants H' and K' of the new equation are given by

$$(8)\quad H' = \phi'(u')\psi'(v')H, \qquad K' = \phi'(u')\psi'(v')K,$$

where ϕ' and ψ' are the first derivatives of ϕ and ψ with respect to their arguments.

Consider the coördinates

$$(9)\quad x_1 = \frac{\partial x}{\partial v} - \frac{\partial\log a}{\partial v}x,$$

of the net N_1 mentioned in the introduction. If we differentiate with respect to u, we get

$$(10)\quad \frac{\partial x_1}{\partial u} - \frac{\partial\log b}{\partial u}x_1 = Hx,$$

a relation confirming the statement of the introduction that the lines joining corresponding points of N and N_1 are tangent to the curves v = const. on N_1. Then if H vanishes equations (9) and (10) reduce the solution of equation (1) to quadratures; also in this case, the surface N_1 degenerates into a curve. But if H does not vanish, we differentiate with respect to v and find that the coördinates x_1 satisfy the equation of Laplace

$$(11)\quad \begin{aligned}\frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial\log aH}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial\log b}{\partial u}\frac{\partial\theta}{\partial v}\\ &+\left(\frac{\partial^2}{\partial u\partial v}\log\frac{b}{a} - \frac{\partial\log aH}{\partial v}\frac{\partial\log b}{\partial u} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c\right)\theta,\end{aligned}$$

which proves that N_1 is also a net, as stated in the introduction. This equation has invariants H_1 and K_1, analogous to H and K, defined by

$$(12)\quad \begin{aligned}H_1 &= -\frac{\partial^2}{\partial u\partial v}\log\frac{a^2H}{b} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c = H - \frac{\partial^2}{\partial u\partial v}\log\frac{aH}{b},\\ K_1 &= -\frac{\partial^2}{\partial u\partial v}\log a + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u} + c = H.\end{aligned}$$

Since frequent use is to be made of the point equation of nets associated with a net having the point equation (1), for the sake of brevity we denote such an equation by the expression

$$[x_i;\ a_i,\ b_i,\ c_i], \tag{13}$$

which means that the coördinates x_i of the net N_i satisfy the equation

$$\begin{aligned} \frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial \log a_i}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b_i}{\partial u}\frac{\partial \theta}{\partial v} \\ &+ \left(\frac{\partial^2 \log c_i}{\partial u \partial v} - \frac{\partial \log a_i}{\partial v}\frac{\partial \log b_i}{\partial u} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c\right)\theta. \end{aligned} \tag{14}$$

Also the net N_i has invariants

$$\begin{aligned} H_i &= -\frac{\partial^2}{\partial u\partial v}\log\frac{a_i}{c_i} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \\ K_i &= -\frac{\partial^2}{\partial u\partial v}\log\frac{b_i}{c_i} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c. \end{aligned} \tag{15}$$

In this notation (11) becomes

$$[x_1;\ aH,\ b,\ b/a], \tag{16}$$

and the effect of the transformation (5) on the point equation is expressed by

$$\left[\frac{x_i}{\lambda};\ \frac{a_i}{\lambda},\ \frac{b_i}{\lambda},\ \frac{c_i}{\lambda}\right]. \tag{17}$$

The minus first Laplace transform, N_{-1}, is the second focal surface of the congruence of tangents to the curves $v =$ const. of N. For, by the definition of its coördinates given in equation (2), the lines joining corresponding points are tangent to N, and the equation obtained by differentiating these coördinates with respect to v and using (1) shows them to be tangent to N_{-1}. The point equation of this net is denoted by

$$[x_{-1};\ a,\ bK,\ a/b]. \tag{18}$$

Consider now the congruences of tangents to the parametric curves of N_1. We have seen that the congruence of tangents to the curves $v =$ const. has N and N_1 as its focal nets; that is, N is the minus first Laplace transform of N_1. This is also obvious as a consequence of equation (10), whose left member is the expression for the coördinates of the minus first Laplace transform of N_1 formed by analogy with (2).

The second focal surface of the congruence of tangents to the curves $u =$ const. of N_1 is the net N_2, the second Laplace transform of N. Its

coördinates

$$x_2 = \frac{\partial x_1}{\partial v} - \frac{\partial \log aH}{\partial v} x_1, \tag{19}$$

or, using (9),

$$x_2 = \frac{\partial^2 x}{\partial v^2} - \frac{\partial \log a^2 H}{\partial v}\frac{\partial x}{\partial v} + \left(\frac{\partial \log a}{\partial v}\frac{\partial \log aH}{\partial v} - \frac{\partial^2 \log a}{\partial v^2}\right)x,$$

satisfy the equation denoted by $[x_2;\ aHH_1,\ b,\ b^2/a^2H]$.

Continuation of this process in both the positive and negative senses gives the nets of the sequence N_r.

The coördinates of the rth Laplace transform N_r are

$$x_r = \frac{\partial x_{r-1}}{\partial v} - \frac{\partial \log aHH_1 \cdots H_{r-2}}{\partial v} x_{r-1}, \tag{20}$$

or, by repeated substitution,

$$x_r = \frac{\partial^r x}{\partial v^r} + A_{r,\,r-1}\frac{\partial^{r-1} x}{\partial v^{r-1}} + \cdots + A_{r,\,0}x. \tag{21}$$

where the $A_{p,\,q}$ are functions of $a,\ H,\ H_1,\ \cdots,\ H_{r-2}$ and their derivatives. The point equation of N_r is

$$\left[x_r;\ aHH_1 \cdots H_{r-1},\ b,\ \frac{b^r}{a^r H^{r-1} H_1^{r-2} \cdots H_{r-2}}\right]. \tag{22}$$

The equations analogous to (10) and (19) are

$$\frac{\partial x_r}{\partial u} = H_{r-1}x_{r-1} + \frac{\partial \log b}{\partial u}x_r; \qquad \frac{\partial x_r}{\partial v} = \frac{\partial \log a_r}{\partial v}x_r + x_{r+1}, \tag{23}$$

and they will be used as formulas for the partial derivatives $\partial x_r/\partial u$ and $\partial x_r/\partial v$.

On the negative side of the sequence, the general Laplace transform N_{-s} has coördinates

$$x_{-s} = \frac{\partial x_{-s+1}}{\partial u} - \frac{\partial \log bKK_{-1} \cdots K_{-s+2}}{\partial u} x_{-s+1},$$

or

$$x_{-s} = \frac{\partial^s x}{\partial u^s} + B_{s,\,s-1}\frac{\partial^{s-1} x}{\partial u^{s-1}} + \cdots + B_{s,\,1}\frac{\partial x}{\partial u} + B_{s,\,0}x, \tag{24}$$

which satisfy the equation

$$\left[x_{-s};\ a, bKK_{-1} \cdots K_{-s+1},\ \frac{a^s}{b^s K^{s-1} \cdots K_{-s+2}}\right]. \tag{25}$$

The formulas corresponding to (23) are

$$\frac{\partial x_{-s}}{\partial u} = x_{-s-1} + \frac{\partial \log b_{-s}}{\partial u}x_{-s}; \qquad \frac{\partial x_{-s}}{\partial v} = \frac{\partial \log a}{\partial v}x_{-s} + K_{-s+1}x_{-s+1}. \tag{26}$$

From (23) and (26) it follows that if H_{r-1} or K_{-s+1} vanishes, the sequence terminates; for the surface X_r or X_{-s} degenerates into a curve. This is a special case of great importance* but it is not before us in this paper.

2. **Periodic Sequences of Laplace.** In the introduction, a periodic sequence was defined as a sequence such that a certain net X_p coincided with the original net X. When this is the case, the coördinates x_p and x must satisfy the relation

$$x_p = \lambda(u, v)x, \tag{27}$$

where λ is a function of u and v at most, and is the same for all n coördinates. The coördinates x_p satisfy the equation denoted by

$$\left[x_p;\ aHH_1 \cdots H_{p-1},\ b,\ \frac{b^p}{a^pH^{p-1}\cdots H_{p-2}\partial}\right], \tag{28}$$

this result being obtained when r in (22) is replaced by p. From (17), (27), and (28), the coördinates x must satisfy

$$\left[x;\ \frac{aHH_1 \cdots H_{p-1}}{\lambda},\ \frac{b}{\lambda},\ \frac{b^p}{a^pH^{p-1}\cdots H_{p-2}\lambda}\right], \tag{29}$$

as well as the fundamental equation (1). Since in every case which we shall consider there are at least three coördinates x, the coefficients of $\partial x/\partial u$, $\partial x/\partial v$, and x in (29) and in (1) must be equal. We have therefore

$$\frac{\partial}{\partial v}\log\frac{aH\cdots H_{p-1}}{\lambda} = \frac{\partial \log a}{\partial v}, \qquad \frac{\partial}{\partial u}\log\frac{b}{\lambda} = \frac{\partial \log b}{\partial u}, \tag{30}$$

$$\frac{\partial^2}{\partial u\partial v}\log\frac{b^p}{a^pH^{p-1}\cdots H_{p-2}\lambda} = 0. \tag{31}$$

From the equations (30), we get

$$\frac{\partial \log\lambda}{\partial v} = \frac{\partial \log HH_1\cdots H_{p-1}}{\partial v}, \qquad \frac{\partial \log \lambda}{\partial u} = 0, \tag{32}$$

and from these

$$\frac{\partial^2}{\partial u\partial v}\log HH_1\cdots H_{p-1} = 0. \tag{33}$$

Using (32) and (33) in (31), we get

$$\frac{\partial^2}{\partial u\partial v}\log\frac{b^p}{a^pH^{p-1}\cdots H_{p-2}} = 0, \tag{34}$$

which can also be obtained immediately from the equality of H, the invariant of (1), and H_p, the corresponding invariant of (28).

* Darboux, l.c., p. 33.

Equation (33) may be integrated, giving

$$HH_1 \cdots H_{p-1} = UV,$$

where U and V are functions of u and v alone respectively.

From equation (8) we recall the effect of the transformation (7) on the invariants H and K. Likewise under this same transformation

$$H_i' = \phi'(u')\psi'(v')H_i.$$

By giving to i values from 0 to $p - 1$ and multiplying, we get

$$H'H_1' \cdots H_{p-1}' = HH_1 \cdots H_{p-1}[\phi'\psi']^p = \overline{U}(u')\overline{V}(v')[\phi'\psi']^p,$$

where $\overline{U}$ and $\overline{V}$ are the transforms of U and V under (7). Hence ϕ and ψ may be determined so that

$$H'H_1' \cdots H_{p-1}' = 1;$$

then from (32), λ equals a constant,* m, since

$$\frac{\partial \log \lambda}{\partial u} = \frac{\partial \log \lambda}{\partial v} = 0.$$

In the remainder of this section, we shall assume that this transformation has been made, dropping primes for convenience.

After this change of variable, there are two necessary conditions for a sequence of Laplace of period p, namely

$$HH_1H_2 \cdots H_{p-1} = 1 \tag{36}$$

and equation (34). To show that these conditions are sufficient, we proceed as follows. Differentiate

$$x_p = mx \tag{37}$$

with respect to u. Using (23), (2), and (37), we find

$$H_{p-1}x_{p-1} = mx_{-1}, \tag{38}$$

which states analytically the fact, evident from geometry, that if N_p coincides with N, then N_{p-1} coincides with N_{-1}. Differentiating the last equation with respect to u, and using (23) and (26), we have

$$H_{p-1}H_{p-2}x_{p-2} + H_{p-1}\frac{\partial \log bH_{p-1}}{\partial u}x_{p-1} = mx_{-2} + m\frac{\partial \log bK}{\partial u}x_{-1}. \tag{39}$$

Now $K = H_{-1}$ and $H_{-1} = H_{p-1}$, since (38) is a transformation of the type (5). The equality of K and H_{p-1} may also be derived from (34) and (36), using the values of these invariants given by (15). Then by (38),

* Tzitzeica, Comptes Rendus, vol. 157 (1913), p. 908.

equation (39) reduces to

$$H_{p-1}H_{p-2}x_{p-2} = mx_{-2}. \tag{40}$$

If we continue this process we have in general,

$$H_{p-1}H_{p-2} \cdots H_{p-i}x_{p-i} = mx_{-i}, \tag{41}$$

or, by (36),

$$x_{p-i} = mHH_1 \cdots H_{p-i-1}x_{-i} \tag{42}$$

and finally

$$x = mx_{-p} \tag{43}$$

showing that N is identical with its minus pth Laplace transform as well as with the pth transform. We observe that this process is reversible, that is, by starting from (43), differentiating with respect to v, and reducing step by step, we may reproduce this same set of equations.

If we refer to (21) and (24) it is evident that the $p + 1$ equations given by (41) when i takes integral values from 0 to p inclusive, are a system of linear partial differential equations of various orders which, with (1), must be satisfied by the coördinates of the fundamental net N of a periodic sequence. From this point of view, let us examine in detail the transformation from equation (38) into (40), as this is entirely typical of the change from any one of (41) into the next. The substitutions for $\partial x_{p-1}/\partial u$ and $\partial x_{-1}/\partial u$ from (23) and (2) first engage our attention. The value of $\partial x_{p-1}/\partial u$ used depends on the definition of x_{p-2} and on the use of the point equation of N_{p-2}. But this point equation is essentially the result of differentiating (1) $p - 2$ times with respect to v, a fact which becomes evident on consideration of the result of substituting the value of x_{p-2} from (21) in the point equation denoted by (22). The value of $\partial x_{-1}/\partial u$ used is merely the definition of the minus first Laplace transform. The rest of the reduction may be based as indicated on the two equations (34) and (36). From these considerations and from the reversibility of the process we conclude that, by virtue of (1), its derivatives, and the conditions (34) and (36), any one of equations (41) or (42) is equivalent to any of the others.

If the period be odd, let us set $p = 2n + 1$, and $i = n$ in (42), so that it becomes

$$x_{n+1} = mHH_1 \cdots H_nx_{-n}.$$

Also by setting $p = 2n + 1$ and $i = n + 1$ in (41), we get

$$x_{-n-1} = \frac{1}{m}H_{2n}H_{2n-1} \cdots H_nx_n.$$

The differential equations to which these are equivalent give values of

$\partial^{n+1}x/\partial v^{n+1}$ and $\partial^{n+1}x/\partial u^{n+1}$ in terms of the $2n + 1$ or p quantities

$$\frac{\partial^n x}{\partial u^n}, \frac{\partial^n x}{\partial v^n}, \frac{\partial^{n-1} x}{\partial u^{n-1}}, \frac{\partial^{n-1} x}{\partial v^{n-1}}, \cdots, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, x.$$

All other derivatives of order $n + 1$ may be obtained in terms of these same p quantities by differentiation of (1). Similarly, when the period is even, let $p = 2n$ and $i = n, n + 1$, giving the two equations

$$x_n = mHH_1 \cdots H_{n-1}x_{-n},$$

$$x_{-n-1} = \frac{1}{m} H_{2n-1}H_{2n-2} \cdots H_{n-1}x_{n-1}.$$

By means of these equations and (1) all derivatives of the nth order but $\partial^n x/\partial u^n$, and all derivatives of higher orders may be expressed in terms of the $2n$ or p quantities

$$\frac{\partial^n x}{\partial u^n}, \frac{\partial^{n-1} x}{\partial u^{n-1}}, \frac{\partial^{n-1} x}{\partial v^{n-1}}, \cdots, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, x.$$

In either case we have a completely integrable system of equations which possesses but p independent solutions. From this result follows the theorem stated by Tzitzeica:

A sequence of Laplace of period p can exist in space of no higher order than $p - 1$.

In particular we note:

The only nets of period three are planar nets.

It will be observed that the conditions (34) and (36) do not involve the constant m. Neither is it involved in the above discussion of the complete integrability of equations (1) and (37). Again, the equations themselves show us that m is a significant constant, that is, one which cannot be reduced to unity by any change of parameter. Then the solutions of the system, which are the coördinates x of our fundamental net N, may be written $x^i\ (u, v; m)$, $i = 1, 2, \cdots, p$.

If we replace m by another constant, m', so that we have the equation $x_p' = m'x'$, instead of (37), this equation forms with (1) another completely integrable system with p independent solutions, which we may call $x'\ (u, v; m')$. A similar set may be obtained for every value of the constant. We may state this result as follows:

If an equation of Laplace be the point equation of a net whose Laplace sequence is periodic of period p, it is the point equation of an infinity of nets having the same property.

3. **Levy sequences. The first Levy sequence.** In equation (3) of the introduction, functions ξ and η are defined as the coördinates of the Levy

transforms of N by means of a solution θ of the point equation (1). For the study of these transforms in connection with the Laplace sequence, it is advantageous to denote the coördinates by $x_{0,1}$ and $x_{-1,0}$, defined by

$$x_{0,1} = \frac{1}{\theta}\begin{vmatrix} \theta & x \\ \theta_1 & x_1 \end{vmatrix}, \qquad x_{-1,0} = \frac{1}{\theta_{-1}}\begin{vmatrix} \theta_{-1} & x_{-1} \\ \theta & x \end{vmatrix},$$

as they indicate by their form that the points given by any parameter values lie on the line joining the points of N and its Laplace transforms with the same parameters. The functions θ_1 and θ_{-1}, defined by

$$\theta_1 = \frac{\partial\theta}{\partial v} - \frac{\partial \log a}{\partial v}\theta; \qquad \theta_{-1} = \frac{\partial\theta}{\partial u} - \frac{\partial \log b}{\partial u}\theta, \tag{44}$$

are called the first and minus first Laplace transforms of θ and are solutions of the point equation of N_1 and N_{-1} respectively. As we have the relations

$$\theta x_{0,1} = -\frac{\partial\theta}{\partial v}\xi, \qquad \theta_{-1}x_{-1,0} = \frac{\partial\theta}{\partial u}\eta,$$

the functions $x_{0,1}$ and $x_{-1,0}$ differ from ξ and η only by factors of proportionality, and consequently are coördinates of the Levy transforms.

We shall accordingly denote the Levy transforms by $N_{0,1}$ and $N_{-1,0}$; their point equations are denoted by

$$\left[x_{0,1};\ \frac{a\theta_1}{\theta},\ b,\ \frac{b}{\theta}\right], \tag{45}$$

and

$$\left[x_{-1,0};\ \frac{a\theta}{\theta_{-1}},\ b,\ \frac{a}{\theta_{-1}}\right].$$

Let the net $N_{r,r+1}$ be defined by its coördinates $x_{r,r+1}$ namely,

$$x_{r,r+1} = \frac{1}{\theta_r}\begin{vmatrix} \theta_r & x_r \\ \theta_{r+1} & x_{r+1} \end{vmatrix}, \tag{46}$$

where r is any positive or negative integer or zero, and where θ_i is formed from θ by (21) and (24) as x_i is formed from x. The order of the subscripts in $N_{r,r+1}$ indicates that the points of these nets are to be considered as situated on the tangents to the curves u = constant of the net N_r; that is, on the line between any net N_r and its positive Laplace transform, N_{r+1}. The application of (23) and (26) to nets of this type leads to the theorem:

Any Laplace transform N_r of a net N has the nets $N_{r-1,r}$ and $N_{r,r+1}$ as its Levy transforms by means of θ_r, the rth Laplace transform of a solution θ of the point equation of N; or, $N_{r,r+1}$ is a Levy transform of N_r by means of θ_r, and of N_{r+1} by means of θ_{r+1}.

Let us express the coördinates of the first Laplace transform of $N_{-1,0}$, following (2) and simplify them. We find

$$\frac{\partial x_{-1,0}}{\partial v} - \frac{\partial}{\partial v}\log\frac{a\theta}{\theta_{-1}}\,x_{-1,0} = x_{0,1},$$

that is, *the Levy transforms of a net by means of the same solution of its point equation are Laplace transforms of one another.*

From the last two theorems $N_{r-1,r}$ and $N_{r,r+1}$ are Laplace transforms of one another for every value of r. Then $N_{r,r+1}$, $(r = \cdots, -2, -1, 0, 1, 2, \cdots)$, is a sequence of Laplace; it will be called the first Levy sequence. In the expressions for the point equations and the formulas for the partial derivatives of the coördinates of the nets of this sequence, it is necessary to distinguish between positive and negative subscripts. If r and s be positive integers, we have

$$\left[x_{r,r+1};\ \frac{aHH_1\cdots H_{r-1}\theta_{r+1}}{\theta_r},\ b,\ \frac{b^{r+1}}{a^rH^{r-1}H_1^{\,r-2}\cdots H_{r-2}\theta_r}\right];$$

$$\left[x_{-s-1,-s};\ \frac{a\theta_{-s}}{\theta_{-s-1}},\ bKK_{-1}\cdots K_{-s+1},\ \frac{a^{s+1}}{b^sK^{s-1}K_{-1}^{\,s-2}\cdots K_{-s+2}\theta_{-s-1}}\right];$$

$$\frac{\partial x_{r,r+1}}{\partial u} = H_{r-1,r}x_{r-1,r} + \frac{\partial\log b}{\partial u}x_{r,r+1},$$

$$\frac{\partial x_{r,r+1}}{\partial v} = \frac{\partial\log a_{r,r+1}}{\partial v}x_{r,r+1} + x_{r+1,r+2};$$

$$\frac{\partial x_{-s-1,-s}}{\partial u} = \frac{K_{-s-1,-s}}{K_{-s}}x_{-s-2,-s-1} + \frac{\partial\log b_{-s-1,-s}}{\partial u}x_{-s-1,-s},$$

$$\frac{\partial x_{-s-1,-s}}{\partial v} = \frac{\partial\log a_{-s-1,-s}}{\partial v}x_{-s-1,-s} + K_{-s+1}x_{-s,-s+1}.$$

4. **The second Levy sequence. Levy sequences of higher orders.** The first Levy sequence is built up from the fundamental sequence of Laplace by the use of a solution θ of (1) and its Laplace transforms. On this Levy sequence which is itself a sequence of Laplace, we may build a second Levy sequence and so on indefinitely.

Let $\theta_{0,1}$ be a solution of equation (45). The Levy transforms of $N_{0,1}$ by means of this solution are the nets $N_{0,2}$ and $N_{-1,1}$, whose coördinates are defined in accordance with (46) as follows,

$$x_{0,2} = \frac{1}{\theta_{0,1}}\begin{vmatrix}\theta_{0,1} & x_{0,1}\\ \theta_{1,2} & x_{1,2}\end{vmatrix}, \qquad x_{-1,1} = \frac{1}{\theta_{-1,0}}\begin{vmatrix}\theta_{-1,0} & x_{-1,0}\\ \theta_{0,1} & x_{0,1}\end{vmatrix},$$

where $\theta_{1,2}$ is the first Laplace transform of $\theta_{0,1}$, and $\theta_{-1,0}$ is its minus

first transform divided by $H_{-1,0}$, a quantity which occurs similarly in $x_{-1,0}$. The point equations of $N_{0,2}$ and $N_{-1,1}$ are denoted by

$$(47) \qquad \left[x_{0,2};\ \frac{a\theta_1\theta_{1,2}}{\theta\theta_{0,1}},\ b,\ \frac{b^2}{\theta\theta_{0,1}}\right], \qquad \left[x_{-1,1};\ \frac{a\theta\theta_{0,1}}{\theta_{-1}\theta_{-1,0}},\ b,\ \frac{ab}{\theta_{-1}\theta_{-1,0}}\right].$$

The same pair of theorems which established the first Levy sequence and the fact that it is a sequence of Laplace are valid here. We denote by $N_{r,r+2}$ and $N_{-s-2,-s}$ the general nets of the second Levy sequence and give their coördinates, namely

$$x_{r,r+2} = \frac{1}{\theta_{r,r+1}} \begin{vmatrix} \theta_{r,r+1} & x_{r,r+1} \\ \theta_{r+1,r+2} & x_{r+1,r+2} \end{vmatrix},$$

$$x_{-s-2,-s} = \frac{1}{\theta_{-s-2,-s-1}} \begin{vmatrix} \theta_{-s-2,-s-1} & x_{-s-2,-s-1} \\ \theta_{-s-1,-s} & x_{-s-1,-s} \end{vmatrix}.$$

Using the second Levy sequence and a solution of (47), a third Levy sequence may be formed. We shall not give the details of this sequence but pass at once to the kth or general sequence. Here the net corresponding to $N_{0,1}$ and $N_{0,2}$ is $N_{0,1}$. Its coördinates and point equation and the accompanying differentiation formulas can be written down by analogy with the corresponding expressions for $N_{0,1}$ and $N_{0,2}$, and their accuracy established by induction. Similar methods may be applied in the study of the other nets of the general sequence. The coördinates of the general net $N_{r,r+k}$ are defined by

$$x_{r,r+k} = \frac{1}{\theta_{r,r+k-1}} \begin{vmatrix} \theta_{r,r+k-1} & x_{r,r+k-1} \\ \theta_{r+1,r+k} & x_{r+1,r+k} \end{vmatrix},$$

where r is any positive or negative integer or zero, and k any positive integer.

In forming the second Levy sequence, we made use of a solution $\theta_{0,1}$ of equation (45); we now investigate the nature of this function. Suppose θ' to be a solution of (1) such that there is no linear relation connecting θ, θ' and the coördinates x. If

$$(52) \qquad \theta_{0,1} = \frac{1}{\theta} \begin{vmatrix} \theta & \theta' \\ \theta_1 & \theta_1' \end{vmatrix},$$

then $\theta_{0,1}$ is a solution of (45). It will now be proved that, conversely, to a solution $\theta_{0,1}$ of (45), not linearly dependent on the coördinates $x_{0,1}$, there corresponds a solution θ' of (1) linearly independent of the x's and of θ. Consider the net $\bar{N}_{0,1}$ as the projection in $(n-1)$ space of a net $\bar{N}_{0,1}$ in n-space whose coördinates are $x_{0,1}^{(1)}, x_{0,1}^{(2)}, \cdots, x_{0,1}^{(n)}, \theta_{0,1}$. Then the congruence G, composed of the lines joining corresponding points

of N and N_1, is the projection of a congruence $\bar{G}$ in n-space conjugate to the net $\bar{N}_{0,1}$. One of the focal nets of this congruence, say $\bar{N}$, projects into the net N. Now the solutions $x^{(i)}$ of (1) are coördinates both of N and of $\bar{N}$ and, with θ, play the same rôle in both spaces in forming the coördinates $x_{0,1}{}^{(i)}$ of $N_{0,1}$ and $\bar{N}_{0,1}$. But in order to form the last coördinates of $\bar{N}_{0,1}$, namely $\theta_{0,1}$, there must be an $(n+1)$st coördinate of $\bar{N}$, a solution of (1) which may be called θ'.

Again the third Levy sequence depends on a solution $\theta_{0,2}$ of (47) for its formation. The argument of the last paragraph then demands as a necessary and sufficient condition for the existence of this solution a second solution $\theta'_{0,1}$ of (45) not linearly dependent on those already obtained. In the same manner, $\theta'_{0,1}$ calls for a third solution, say θ'', of (1) not linearly dependent on the solutions already used, such that

$$\theta'_{0,1} = \frac{1}{\theta}\begin{vmatrix} \theta & \theta'' \\ \theta_1 & \theta_1'' \end{vmatrix}.$$

The final effect of this argument is to base the kth or general sequence on k solutions, $\theta, \theta', \cdots, \theta^{(k-1)}$ of (1) such that there is no linear relation between them and the x's.

For further developments, we must prove, as a lemma, a property of determinants. Consider the general determinant of the nth order

$$D = |a_{lm}|, \qquad l, m = 1, 2, \cdots n.$$

Subtract from each element of the ith row the product of the corresponding element of the $(i-1)$st row by $a_{i,1}/a_{i-1,1}$, $(i = n, n-1, \cdots, 2,)$ and develop the result by the elements of the first column. We have

$$D = a_{1,1}\begin{vmatrix} \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} & \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} & \cdots & \frac{1}{a_{1,1}}\begin{vmatrix} a_{1,1} & a_{1,n} \\ a_{2,1} & a_{2,n} \end{vmatrix} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a_{n-1,1}}\begin{vmatrix} a_{n-1,1} & a_{n-1,2} \\ a_{n,1} & a_{n,2} \end{vmatrix} & & \cdots & \frac{1}{a_{n-1,1}}\begin{vmatrix} a_{n-1,1} & a_{n-1,n} \\ a_{n,1} & a_{n,n} \end{vmatrix} \end{vmatrix} = a_{1,1}\Delta$$

where Δ is of order $n-1$, so that

$$\Delta = \frac{1}{a_{11}} D. \tag{53}$$

The coördinates of $N_{0,2}$ are

$$x_{0,2} = \frac{1}{\theta_{0,1}}\begin{vmatrix} \theta_{0,1} & x_{0,1} \\ \theta_{1,2} & x_{1,2} \end{vmatrix}.$$

Using $r = 0, 1$ in (46) and the analogous expressions for $\theta_{0,1}$ and $\theta_{1,2}$ the coördinates $x_{0,2}$ become determinants of the form of Δ. On applying the

property expressed in (53) to them, we find

$$x_{0,2} = \frac{1}{\theta\theta_{0,1}} \begin{vmatrix} \theta & \theta' & x \\ \theta_1 & \theta_1' & x_1 \\ \theta_2 & \theta_2' & x_2 \end{vmatrix} \equiv \frac{1}{\theta\theta_{0,1}} \left| \theta\theta_1' x_2 \right|,$$

the latter expression being an abbreviated form in which only the elements of the main diagonal are shown.

Consider $x_{0,k}$, the coördinates of the net $N_{0,k}$. By definition

$$x_{0,k} = \frac{1}{\theta_{0,k-1}} \begin{vmatrix} \theta_{0,k-1} & x_{0,k-1} \\ \theta_{1,k} & x_{1,k} \end{vmatrix}.$$

Then by (53)

$$x_{0,k} = \frac{1}{\theta_{0,k-2}\theta_{0,k-1}} \left| \theta_{0,k-2}\, \theta_{1,k-1}'\, x_{2,k} \right|,$$

and by its repeated use

$$x_{0,k} = \frac{1}{\theta\theta_{0,1}\theta_{0,2} \cdots \theta_{0,k-1}} \left| \theta\theta_1'\theta_2'' \cdots \theta_{k-1}^{(k-1)} x_k \right|.$$

As this method of exhibiting the coördinates of the nets of the Levy sequences is a purely algebraic matter, we have at once,

$$(54) \qquad x_{r,r+k} = \frac{1}{\theta_r\theta_{r,r+1} \cdots \theta_{r,r+k-1}} \left| \theta_r\theta_{r+1}' \cdots \theta_{r+k-1}^{(k-1)} x_{r+k} \right|,$$

where r may be any positive or negative integer, or zero. We shall call the determinant in the above equation $X_{r,r+k}$; a determinant like it but for the last column, in which the Laplace transforms of x are replaced by those of $\theta^{(k)}$, a $(k+1)$st solution of (1), we shall call $\Theta_{r,r+k}$. Then

$$(55) \qquad \theta_{r,r+k} = \frac{1}{\theta_r\theta_{r,r+1} \cdots \theta_{r,r+k-1}} \Theta_{r,r+k}.$$

From (54) and (55), we have

$$(56) \qquad \theta_r\theta_{r,r+1} \cdots \theta_{r,r+k-1} x_{r,r+k} = X_{r,r+k},$$

and

$$(57) \qquad \theta_r\theta_{r,r+1} \cdots \theta_{r,r+k} = \Theta_{r,r+k}.$$

These equations are valid for any integral value of r, and for any positive integral value of k. If we replace k in (57) by $k-1$ and use the result in (56) and (57), we get

$$(58) \qquad \Theta_{r,r+k-1} x_{r,r+k} = X_{r,r+k}$$

and

$$\Theta_{r,r+k-1}\theta_{r,r+k} = \Theta_{r,r+k}.$$

Since the $X_{r,r+k}$ are proportional to the $x_{r,r+k}$, the former may serve equally well as homogeneous coördinates of the nets $N_{r,r+k}$.

In the preceding paragraph we have used solutions θ linearly independent of the coördinates x. The following theorem states the situation under the opposite condition.

If a solution θ of the equation (1) *used in the formation of any Levy sequence be linearly dependent on the coördinates x, all the nets of this Levy sequence lie in $(p - 2)$ space; if i such solutions be used, in $(p - i - 1)$ space.*

For, suppose $\theta = \sum_{i=1}^{i=n} g^{(i)}x^{(i)}$ where the $g^{(i)}$ are constants not all zero; then $\theta_k = \sum_{i=1}^{i=n} g^{(i)}x_k^{(i)}$ for every k. Now using these values of the Laplace transforms of θ, we have

$$\sum_{i=1}^{n} g^{(i)}X^{(i)}_{r,\,r+k} = |\theta_r \theta'_{r+1} \cdots \theta^{(k-1)}_{r+k-1} \Sigma g^{(i)}x_{r+k}| = 0,$$

since the first and last columns are identical, that is, the coördinates of all nets of the sequence $N_{r,\,r+k}$ satisfy the equation of the hyperplane $\sum_{i=1}^{i=n} g^{(i)}z^{(i)} = 0$, where the $z^{(i)}$ are current coördinates. We observe that if all the $g^{(i)}$ but one, say $g^{(j)}$, are zero, the nets lie in the coördinate hyperplane $x^{(j)} = 0$. If $\theta' = \sum_{i=1}^{i=n} h^{(i)}x^{(i)}$, then by the above argument, the nets of the sequence lie also in the hyperplane $\sum_{i=1}^{i=n} h^{(i)}z^{(i)} = 0$. Thus they lie in the intersection of two hyperplanes, or in space of order $p - 3$. This proof may be extended to the case stated in the theorem.

Consider the coördinates of the net $N_{-s,\,r}$; $r, s \geq 0$, in the form

$$X_{-s,\,r} = |\theta_{-s}\theta'_{-s+1} \cdots \theta^{(k-1)}_{r-1} x_r|,$$

where we have written only the elements of the main diagonal. For each of the Laplace transforms occurring in these determinants substitute their values as linear functions of the derivatives of the θ's and the x's from (21) and (24). By suitable operations on the rows, the determinants may then be reduced to the form

$$X_{-s,\,r} = \left| \frac{\partial^s \theta}{\partial u^s} \frac{\partial^{s-1}\theta'}{\partial u^{s-1}} \cdots \theta^{(s)} \cdots \frac{\partial^{r-1}\theta^{(k-1)}}{\partial v^{r-1}} \frac{\partial^r x}{\partial v^r} \right|.$$

In this form the identity of the $X_{-s,\,r}$ with the coördinates of the derived nets of higher order k as defined by Tzitzeica is obvious. There are $k + 1$ derived nets of order k; for example, the Levy transforms $N_{-1,\,0}$ and $N_{0,\,1}$ are the derived nets of the first order; the nets $N_{-2,\,0}$, $N_{-1,\,1}$, and $N_{0,\,2}$ are the derived nets of order two, and so for higher orders. We note especially that $N_{-1,\,1}$ is *the derived net* of N depending on θ and θ' in the restricted use of that term; N, in turn, is *the derivant net* of N. Extending this term, we say that N is a derivant net of all nets $N_{-s,\,r}$; $r, s \geq 0$.

5. **Periodic Levy sequences.** If a sequence of Laplace is of period p, and in $(p - 1)$ space, we shall now develop certain conditions under which its Levy sequences have this same period. If the first Levy sequence is

to be periodic, we must have

$$x_{p,\,p+1} = \lambda x_{0,\,1},$$

where λ is an undetermined factor of proportionality. By the use of (37) and the value of x_{p+1} found by differentiating (37) with respect to v, we obtain

$$\frac{m}{\theta_p}\begin{vmatrix} \theta_p & x \\ \theta_{p+1} & x_1 \end{vmatrix} = \frac{\lambda}{\theta}\begin{vmatrix} \theta & x \\ \theta_1 & x_1 \end{vmatrix},$$

that is,

$$x_1(m - \lambda) - x\left(\frac{m\theta_{p+1}}{\theta_p} - \frac{\lambda\theta_1}{\theta}\right) = 0.$$

Now there are at least three coördinates x, and accordingly the coefficients of x and x_1 must be zero. We have $m = \lambda$, and $\theta_{p+1}\,\theta_p = \theta_1\,\theta$, which, because of the definition of θ_{p+1} and θ_1, and equation (36), becomes

$$\frac{\partial}{\partial v}\log\frac{\theta_p}{\theta} = 0. \tag{59}$$

Let us now differentiate $x_{p,\,p+1} = mx_{0,\,1}$ with respect to u. By applying formulas already derived for such derivatives, we get

$$x_{p-1,\,p} = mx_{-1,\,0}.$$

In this case, using (37) and (38) and expanding, we have $H_{p-1}\theta_{p-1}\,\theta_p = \theta_{-1}\,\theta$. Then (2) and (23) give

$$\frac{\partial}{\partial u}\log\frac{\theta_p}{\theta} = 0. \tag{60}$$

In consequence of (59) and (60), we see that

$$\theta_p = C_1\theta,$$

where C_1 is a constant whose value is to be studied further. To show that under this condition the nets $N_{p-i,\,p-i+1}$ and $N_{-i,\,-i+1}$ are identical, consider the coördinates

$$x_{p-1,\,p-i+1} = \frac{1}{\theta_{p-i}}\begin{vmatrix} \theta_{p-i} & x_{p-i} \\ \theta_{p-i+1} & x_{p-i+1} \end{vmatrix}.$$

Since (41) and (42) are true for θ if m be replaced by C_1, we have

$$x_{p-i,\,p-i+1} = mHH_1\cdots H_{p-i}x_{-i,\,-i+1}.$$

For the second Levy sequence also to be periodic, it is necessary and sufficient that $\theta_{0,\,1}$ the solution of (45) by which the second Levy sequence is formed from the first, shall be such that

$$\theta_{p,\,p+1} = C_2\theta_{0,\,1}.$$

By referring to equation (52), we see that this will be the case if $\theta_p' = C_1'\theta'$. In general, we conclude that if the $(k-1)$st sequence is periodic, the necessary and sufficient condition for the kth sequence to be periodic, is that

$$\theta_{p,\ p+k-1} = C_k\theta_{0,\ k-1}$$

and that this condition will be fulfilled if $\theta, \theta', \cdots \theta^{(k-1)}$ are such that their pth Laplace transforms are constant multiples of them. Evidently under these last conditions not only is the kth Levy sequence periodic, but also all the sequences of order less than k.

The disposition of the constant multipliers in various ways leads to some interesting results. By the last theorem of section 2, there are p solutions θ for every value of the constant occurring in (37). First, let us suppose that C_1 is equal to m. Then θ must be a linear combination of the x's and therefore the theorem of section 4 applies and the nets of this periodic Levy sequence lie space of order $p-2$. This result was noted by Tzitzeica. For sequences of higher orders, it may be generalized into the following theorem:

If a sequence of Laplace of period p lie in $(p-1)$ space, and has coördinates such that $x_p = mx$, and if a Levy sequence of order k, periodic or not, based on this sequence of Laplace, be formed by the use of k solutions θ of the original point equation, of which one is such that $\theta_p = m\theta$, the nets of this Levy sequence lie in space of order $p-2$; if i such solutions be used, the Levy sequence is in space of order $p-i-1$.

To prove this theorem, we need first to recall that there are but p solutions of the system of partial differential equations satisfied by the coördinates of a periodic sequence of Laplace; therefore, if $\theta_p = m\theta$, θ is a linear function of the coördinates x. The proof is then completed by the application of the last theorem of section 4.

Consider now the Levy sequences which can be formed on a periodic sequence of Laplace by the use of the set of p solutions $\theta, \theta', \cdots, \theta^{(p-1)}$ such that $\theta_p^{(i)} = m'\theta^{(i)}$, $m' \neq m$. There will be p periodic first Levy sequences, $p(p-1)/2$ periodic second Levy sequences, in general, as many of the kth order as the number of combinations of p things taken k at a time and finally, one periodic sequence of the pth order. It will now be shown that this pth sequence coincides with the original sequence of Laplace. For, consider the coördinates of the net $X_{0,\ p}$, namely, $X_{0,\ p}$. We have

$$X_{0,\ p} = \begin{vmatrix} \theta & \theta' & \cdots & \theta^{(p-1)} & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_p & \theta_p' & \cdots & \theta_p^{(p-1)} & x_p \end{vmatrix} = \begin{vmatrix} \theta & \theta' & \cdots & \theta^{(p-1)} & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m'\theta & m'\theta' & \cdots & m'\theta^{(p-1)} & mx \end{vmatrix}.$$

Subtracting m' times the first row from the last, then

$$X_{0,\,p} = (m - m')\Theta_{0,\,p-1}x,$$

so that the coördinates $X_{0,\,p}$ are proportional to the x's. In general the coördinates $X_{r,\,r+p}$ of the net $N_{r,\,r+p}$ are determinants such that in each the elements of its last row may be made all zero but the last, which will be a constant multiple of x_r. Therefore $X_{r,\,r+p}$ is proportional to x_r, and the nets $N_{r,\,r+p}$ coincide with the original Laplace sequence.

6. **Nets in relation T and their Laplace transforms.** In the introduction a geometric definition of the relation T was given; Eisenhart has shown that, if $\bar{N}$ be a net in relation T with N, their analytic relation is expressed in the statement that the homogeneous coördinates $\bar{x}$ of $\bar{N}$ may be obtained from quadratures of the form

$$\frac{\partial \bar{x}}{\partial u} = \tau \frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial \bar{x}}{\partial v} = \sigma \frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{61}$$

where θ is a solution of (1) different from any x. The factors τ and σ are not entirely arbitrary for the conditions of integrability of (61) show that they must be solutions of the equations

$$\frac{\partial \tau}{\partial v} = (\sigma - \tau)\frac{\partial}{\partial v}\log\frac{a}{\theta}, \qquad \frac{\partial \sigma}{\partial u} = (\tau - \sigma)\frac{\partial}{\partial u}\log\frac{b}{\theta}, \tag{62}$$

or their equivalents,

$$\frac{\partial}{\partial v}\log\frac{a\tau}{\theta} = \frac{\sigma}{\tau}\frac{\partial}{\partial v}\log\frac{a}{\theta}, \qquad \frac{\partial}{\partial u}\log\frac{b\sigma}{\theta} = \frac{\tau}{\sigma}\frac{\partial}{\partial u}\log\frac{b}{\theta}.$$

In connection with the derivation of the integrability conditions of (61) it is readily shown that the net $\bar{N}$ has the point equation

$$\frac{\partial^2 \bar{x}}{\partial u \partial v} = \frac{\partial}{\partial v}\log\frac{a\tau}{\theta}\frac{\partial \bar{x}}{\partial u} + \frac{\partial}{\partial u}\log\frac{b\sigma}{\theta}\frac{\partial \bar{x}}{\partial v}.$$

For an equation of this special type in which the term involving $\bar{x}$ is missing we shall use a symbol similar to (13) except that the last of the quantities within the brackets is omitted to indicate that the term in $\bar{x}$ is lacking. Thus, the point equation of $\bar{N}$ will be denoted by

$$\left[\bar{x};\ \frac{a\tau}{\theta},\ \frac{b\sigma}{\theta}\right]. \tag{63}$$

Using the fact that θ is a solution of (1), the invariants of $\bar{N}$ have the values

$$\bar{H} = H - \frac{\partial^2 \log \tau}{\partial u \partial v}, \qquad \bar{K} = K - \frac{\partial^2 \log \sigma}{\partial u \partial v}. \tag{64}$$

From these developments, it appears that the determination of a net in relation T with N depends on a solution of (1), a pair of functions τ and σ which satisfy (62), and the quadratures (61). Eisenhart has shown that the problem may be given another aspect by the introduction of a function ϕ, defined by the equation

$$\tau - \sigma = \phi\theta. \tag{65}$$

By differentiation and the use of (62) and (1) it may be shown that ϕ is a solution of the equation denoted by

$$\left[\phi;\ \frac{1}{a},\ \frac{1}{b},\ \frac{1}{ab}\right]. \tag{66}$$

But this is the adjoint of (1). Accordingly, the problem is reduced to the finding of a solution of (1) and a solution of its adjoint equation, and two sets of quadratures, namely

$$\begin{aligned} \frac{\partial \tau}{\partial u} &= \phi\theta \frac{\partial \log b\phi}{\partial u}, & \frac{\partial \tau}{\partial v} &= -\phi\theta \frac{\partial}{\partial v} \log \frac{a}{\theta}, \\ \frac{\partial \sigma}{\partial u} &= \phi\theta \frac{\partial}{\partial u} \log \frac{b}{\theta}, & \frac{\partial \sigma}{\partial v} &= -\phi\theta \frac{\partial \log a\phi}{\partial v}, \end{aligned} \tag{67}$$

which follow from (65) and (62), and (61).

A discussion of the effect on the net $\bar{N}$ of the arbitrary constants arising from these quadratures is in order at this time. If $\bar{x}$ be the coördinates of the net $\bar{N}$ when the additive constant c to τ and σ is set equal to zero, then for any other value of c, the coördinates of the T transform become $\bar{x} + cx/\theta$. This point is on the line joining corresponding points of N and $\bar{N}$. Consequently, we may say that the variation of this constant leaves the conjugate congruence of the transformation unchanged but moves the points of the net along the lines of this congruence.

Again if $\bar{x}^{(i)}$ and $\bar{x}^{(i)} + c_i$ are the coördinates of nets obtained by different values of the constant of integration in (61), the line of intersection of the tangent planes to the nets is the same for all values of c_i. This is a result of equation (61) since the coördinates of the Levy transforms of N by means of θ may be taken as $\partial(x/\theta)/\partial u$ and $\partial(x/\theta)/\partial v$. The totality of such lines of intersection, or the joins of corresponding points of the Levy transforms form a congruence which has been termed by Guichard* the harmonic congruence of the transformation. We may say then, that the variation of the constant arising from (61) leaves the harmonic congruence of the transformation unchanged. Conversely, all nets harmonic to this congruence are so determined, since it has been

* Guichard, Annales de l'École Normale Sup., 3e Série, t. 14 (1897), p. 483.

shown by Eisenhart* that two nets harmonic to a congruence are in relation T.

Now if x_1 and θ_1 be the first Laplace transforms of x and θ, the coördinates of $\bar{N}_1$, a T transform of N_1 will be given by quadratures similar to (61), namely,

$$\frac{\partial \bar{x}_1}{\partial u} = \tau_1 \frac{\partial}{\partial u}\left(\frac{x_1}{\theta_1}\right), \qquad \frac{\partial \bar{x}_1}{\partial v} = \sigma_1 \frac{\partial}{\partial v}\left(\frac{x_1}{\theta_1}\right). \tag{68}$$

The integrability conditions of this quadrature give equations for τ_1 and σ_1 analogous to (62),

$$\frac{\partial \tau_1}{\partial v} = (\sigma_1 - \tau_1)\frac{\partial}{\partial v}\log\frac{aH}{\theta_1}, \qquad \frac{\partial \sigma_1}{\partial u} = (\tau_1 - \sigma_1)\frac{\partial}{\partial u}\log\frac{b}{\theta_1}$$

and we also find that the point equation of $\bar{N}_1$ is denoted by

$$\left[\bar{x}_1;\ \frac{aH\tau_1}{\theta_1},\ \frac{b\sigma_1}{\theta_1}\right]. \tag{69}$$

In order to determine the relation between τ_1, σ_1 and τ, σ, we proceed as follows. If $\bar{N}_1$ is the first Laplace transform of $\bar{N}$, the invariants $\bar{H}_1$ and $\bar{H}$ of these nets should be related as are the invariants H_1 and H in (12). Forming the corresponding relation, we have

$$\bar{H}_1 = -\frac{\partial^2}{\partial u \partial v}\log\frac{a\tau\bar{H}}{b\sigma} + \bar{H}.$$

On reducing this equation by the use of (69), (64), and (12), we find that

$$\frac{\partial^2}{\partial u \partial v}\log\tau_1 = \frac{\partial^2}{\partial u \partial v}\log\frac{\tau^2\bar{H}}{\sigma H}. \tag{70}$$

Similar reckoning performed with $\bar{K}_1$ and $\bar{K}$ shows that

$$\frac{\partial^2 \log\sigma_1}{\partial u \partial v} = \frac{\partial^2 \log\tau}{\partial u \partial v}. \tag{71}$$

Now the equations denoted by (63) and (69) are of the form which must be satisfied by the non-homogeneous coördinates of a net, and this suggests that the same relation may hold between the coördinates of $\bar{N}_1$ and $\bar{N}$ that holds in the non-homogeneous case. This is shown to be true, for if we substitute

$$\bar{x}_1 = \bar{x} - \frac{1}{\dfrac{\partial}{\partial v}\log\dfrac{a\tau}{\theta}}\frac{\partial \bar{x}}{\partial v},$$

* A result as yet unpublished.

and

$$\tau_1 = \frac{\tau^2 \overline{H}}{\sigma H}, \qquad \sigma_1 = \tau,$$

—particular solutions of (70) and (71)—and the values of x_1 and θ_1 given by (2) and (44) in equations (68), they are identically true. We have proved then that the T transforms of a net N and its first Laplace transform whose coördinates are obtained from the quadratures (61) and (68), where θ_1 is the first Laplace transform of θ, and where τ_1 and σ_1 have the above values, are Laplace transforms of one another.

We find that the difference $\tau_1 - \sigma_1$, when reduced by the use of (64) and (67), is equal to $- \phi_{-1}\, \theta_1/H$, where

$$\phi_{-1} = \frac{\partial \phi}{\partial u} + \frac{\partial \log b}{\partial u} \phi,$$

following (2) and (66). Now ϕ_{-1} satisfies the equation denoted by

$$\left[\phi_{-1}; \frac{1}{a}, \frac{H}{b}, \frac{1}{a^2}\right], \tag{72}$$

and ϕ_{-1}/H, because of (17), satisfies

$$\left[\frac{\phi_{-1}}{H}; \frac{1}{aH}, \frac{1}{b}, \frac{1}{a^2 H}\right].$$

But this equation is the adjoint of (16); so that we have the net $\overline{N}_1$ based on a solution of the adjoint of the point equation of N_1 which is proportional to the minus first Laplace transform of ϕ.

In general, we have that, if $\overline{N}_r$ is a T transform of N_r, the rth Laplace transform of N, whose coördinates are given by the quadratures

$$\frac{\partial \bar{x}_r}{\partial u} = \tau_r \frac{\partial}{\partial u}\left(\frac{x_r}{\theta_r}\right), \qquad \frac{\partial \bar{x}_r}{\partial v} = \sigma_r \frac{\partial}{\partial v}\left(\frac{x_r}{\theta_r}\right), \tag{73}$$

then the nets $\overline{N}_r$ ($r = 0, 1, 2, \cdots$), form a sequence of Laplace. In this general case, we find, as in the particular cases we have considered, that ϕ'_{-r}, defined by

$$\phi'_{-r} = \frac{\tau_r - \sigma_r}{\theta_r},$$

is a solution of the adjoint of the point equation of N_r, which is denoted by

$$\left[\phi'_{-r}; \frac{1}{aHH_1 \cdots H_{r-1}}, \frac{1}{b}, \frac{b^{r-1}}{a^{r+1} H^r H_1^{r-1} \cdots H_{r-1}}\right]. \tag{74}$$

But using equation (25) we may denote the equation of Laplace satisfied

by ϕ_{-r}, defined as

$$\phi_{-r} = \frac{\partial \phi_{-r+1}}{\partial u} + \frac{\partial}{\partial u} \log \frac{b}{HH_1 \cdots H_{r-1}} \phi_{-r+1}$$

by

$$\left[\phi_{-r};\ \frac{1}{a},\ \frac{HH_1 \cdots H_{r-1}}{b},\ \frac{b^{r-1}}{a^{r+1}H^{r-1} \cdots H_{r-2}} \right]. \tag{75}$$

But the quantity $HH_1 \cdots H_{r-1}\phi'_{-r}$, because of (74) and (17) also satisfies this equation, and therefore we have

$$\frac{\tau_r - \sigma_r}{\theta_r} = \phi'_{-r} = \frac{\phi_{-r}}{HH_1 \cdots H_{r-1}}.$$

Therefore the quadratures to determine τ_r and σ_r, corresponding to (67) are

$$\begin{aligned}
\frac{\partial \tau_r}{\partial u} &= \frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial u} \log \frac{b\phi_{-r}}{HH_1 \cdots H_{r-1}}, \\
\frac{\partial \tau_r}{\partial v} &= -\frac{\theta_r \phi_r}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial v} \log \frac{aH \cdots H_{r-1}}{\theta_r}, \\
\frac{\partial \sigma_r}{\partial u} &= \frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial u} \log \frac{b}{\theta_r}, \\
\frac{\partial \sigma_r}{\partial v} &= -\frac{\theta_r \phi_{-r}}{HH_1 \cdots H_{r-1}} \frac{\partial}{\partial v} \log a\phi_{-r}.
\end{aligned} \tag{76}$$

7. **Periodic sequences of T transforms.** As a preliminary step in the question of the periodicity of the T transforms, we investigate the adjoint equation of (1) when (34) and (36) are satisfied. These equations are necessary and sufficient conditions for a periodic sequence of Laplace whose coördinates satisfy (1) and (37). Now if the invariants of (66) are formed, it is found that they are the same as H and K but are interchanged. Similarly the invariants of (72) are those of (16), that is, H_1 and K_1, but interchanged; and in general, the invariants of (75) are H_r and K_r interchanged. We also notice that in working with (72), a and b are replaced by $1/a$ and $1/b$, respectively. Then the conditions on the coefficients of (66) which assure solutions such that

$$\phi = n\phi_{-p}$$

are equivalent to (34) and (36), and we may state the following theorem:

If an equation of Laplace has periodic solutions, so has its adjoint.

Suppose that the fundamental sequence is periodic of period p, and that $x_p = mx$, the conditions (34) and (36) being satisfied. Let ϕ, the solution of the adjoint equation of (1) which determines the quadratures

(67) and (61), be such that $\phi = n\phi_{-p}$; also let the solution θ of (1) involved in these quadratures be such that $\theta_p = m'\theta$. Then if we set $r = p$ the quadratures (73) and (76) which determine a pth Laplace transform of $\bar{N}$ become

$$\frac{\partial\tau_p}{\partial u} = \frac{m'}{n}\theta\phi\frac{\partial}{\partial u}\log b\phi, \qquad \frac{\partial\tau_p}{\partial v} = -\frac{m'}{n}\theta\phi\frac{\partial}{\partial v}\log\frac{a}{\theta},$$
$$\frac{\partial\sigma_p}{\partial u} = \frac{m'}{n}\theta\phi\frac{\partial}{\partial u}\log\frac{b}{\theta}, \qquad \frac{\partial\sigma_p}{\partial v} = -\frac{m'}{n}\theta\phi\frac{\partial}{\partial v}\log a\phi, \tag{79}$$

and

$$\frac{\partial\bar{x}_p}{\partial u} = \frac{m}{n}\tau\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial\bar{x}_p}{\partial v} = \frac{m}{n}\sigma\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{80}$$

which differ from (67) and (61) only by constant factors. Then the coördinates $\bar{x}_p$ can differ from $\bar{x}$ only by a constant factor arising from the factors appearing in (79) and (80), or by an additive constant from the final quadratures. But these additive constants are entirely arbitrary, and since we are dealing with homogeneous coördinates the factor is immaterial.

We have, now, the following theorem: *Let* (1) *be the point equation of the fundamental net N of a sequence of Laplace of period p, whose coördinates x are such that $x_p = mx$; if θ be a solution of* (1) *such that $\theta_p = m'\theta$, and if ϕ be a solution of the adjoint equation of* (1) *such that $\phi = n\phi_{-p}$, then each T transform of N determined by quadratures from θ and ϕ is the fundamental net of a sequence of Laplace of period p; moreover, each of the nets of these sequences is a T transform of the corresponding net of the original sequence.*

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ON THE TRANSFORMATION OF CONVEX POINT SETS.*

By J. L. Walsh.

It is the primary purpose of this note to prove that a one-to-one point transformation of the plane which transforms every convex point set into a convex point set is a collineation. It is also proved that a regular plane curve which remains convex under all circular transformations of the plane is a circle or circular arc. These results are then extended to space of three dimensions. Throughout the paper we consider geometry in the real domain, but as Professor C. L. Bouton pointed out to me, the discussion holds with minor changes for complex geometry.

§ 1. A point set is said to be *convex* when and only when any two points of the set are joined by a line segment (finite or infinite) consisting entirely of points of the set. According to this definition we shall prove

Theorem I. *A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex point set into a convex point set is that it be a collineation.*

This theorem tacitly assumes the convention, usual in the geometry of the collineation, of a line of points at infinity.† The naturalness of this convention in the present case will appear later.

The sufficiency of the condition of Theorem I is obvious. A collineation transforms lines into lines, line segments into line segments, and hence every convex point set into a convex point set. We proceed to prove the necessity of the condition.

If any two points of the plane are denoted by A' and B', which are the transforms by the transformation T considered of points A and B respectively, we shall prove that any point X' collinear with A' and B' is the transform by T of a point X collinear with A and B. For convenience in phraseology suppose that A, B, A', and B' are finite points. Suppose X', collinear with A' and B', to be the transform by T of a point X not collinear with A and B. The finite segment AB is transformed by T into a point set which includes all points of either the finite or the infinite

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† Some writers define a point set as convex when and only when the finite line segment which joins any two points of the set consists wholly of points of the set. According to this definition, no convention is necessary regarding points at infinity; the following theorem is then analogous to Theorem I and is easily proved by the methods used in the present paper:

A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex point set into a convex point set is that it be an affine transformation.

segment $A'B'$, and X' cannot lie on that particular segment $A'B'$. Hence it is possible to determine a point Y on the finite segment AB such that Y is transformed by T into a point Y' collinear with A' and B', and such that A' and B' separate X' and Y'. Consider now the finite segment XY, a convex point set. This is transformed by T into a point set which includes neither A' nor B', which can include neither the finite nor infinite segment $X'Y'$, and hence which is not convex.

We have thus proved that any three collinear points A', B', X' are the transforms by T of three collinear points A, B, X. It follows immediately from the theorem of Darboux cited below that the inverse of T is a collineation, so T itself is a collineation and the proof of Theorem I is complete.

We have used here the following theorem:

*A one-to-one point transformation of the plane or of space such that collinear points in one configuration correspond to collinear points in the other configuration is a collineation.**

Our convention of a line of points at infinity now appears as entirely natural in connection with Theorem I and hence in the definition of convexity. Without a similar convention, it is not true that all collineations are one-to-one transformations. Without our convention, it is not true that all collineations transform straight lines into straight lines; without it the theorem of Darboux is no longer true. We have therefore introduced this convention to avoid the long circumlocutions which would otherwise be necessary.

It is worth while to notice that the proof of the necessity of the condition of Theorem I uses merely the fact that T transforms every line segment into a convex point set; a correspondingly more general statement of the theorem can be made.

§ 2. Instead of considering convex point sets as in § 1, we shall now consider convex curves. A curve (whether closed or not) is said to be *convex* when and only when it is regular† and no three points of the curve

* Darboux, Mathematische Annalen, vol. XVII (1880), pp. 55–61; p. 59. See also Swift, Bulletin of the American Mathematical Society, (2) vol. X (1903–1904), pp. 247, 361.

Under the hypothesis of this theorem it is easy to prove that an entire straight line in either configuration corresponds to an entire straight line in the other configuration. Darboux pointed out that it was not necessary to suppose the transformation continuous in order to prove it a collineation.

We state explicitly that although the theorem of Darboux and the necessity of the condition of Theorem I are both concerned with the nature of a transformation, neither theorem makes any explicit hypothesis regarding the inverse transformation except its existence and one-to-one-ness.

† For the definition of a regular curve, see Osgood, Funktionentheorie, p. 51. The definition of a regular surface is precisely analogous.

When we consider (as here) transformations of the plane which do not transform every infinite point into an infinite point, the definition of a regular curve is to be revised so that every regular

are collinear unless they lie on a line segment which is an arc of the curve. We shall prove

THEOREM II. *A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex curve into a convex curve is that it be a collineation.*

A collineation transforms straight lines into straight lines, line segments into line segments, and hence every convex curve into a convex curve. The sufficiency of the condition is thus obvious. It will be noticed that the necessity of the condition does not follow from Theorem I or from the theorem indicated at the close of § 1. And neither of those results follows from the necessity of the condition of Theorem II; different proofs are necessary. We shall show that a transformation T which transforms every convex curve into a convex curve transforms every straight line into a straight line or a line segment.

Suppose a straight line C to be transformed by T into a curve C' other than a straight line or line segment. Choose a line l' which has two points M' and N' in common with C' yet which do not lie on a segment of l' which is an arc of C'. Such a choice is evidently possible, since C' is a regular curve. Let P' be any point of l' not on C', and denote by P the point which is transformed into P' by T. Denote by M and N the points which are transformed by T into M' and N'. The curve composed of the segments PM and MN is convex, but is transformed by T into a curve which is not convex.

We ave proved that T transforms every line into a line or line segment. Then T transforms collinear points into collinear points and hence is a collineation.

§ 3. It is not uninteresting to consider groups of transformations other than collineations, and to determine what convex curves remain convex under all transformations of the group. We shall prove:

THEOREM III. *A necessary and sufficient condition that a convex curve remain convex under all circular transformations is that it be a circle or circular arc.**

The sufficiency of the condition is obvious—every circle or circular arc is transformed into a circle or circular arc, which is convex. To prove the necessity of the condition, let us choose three points on the curve C considered which do not lie on a circular arc that is part of C. Such

curve is transformed by a collineation into a regular curve. Cf. Osgood, l.c., pp. 324, 150. The definition of a regular surface is of course to be revised accordingly.

Doubtless Theorems II, III, V, and VI remain true if in the definition of convexity the requirement of regularity is replaced by a suitable less stringent requirement.

* The term *circle* here includes straight line, *circular arc* includes segment of a line. The corresponding convention is not made for the sphere and plane.

choice is possible when and only when C itself is not a circle or circular arc. Transform the circle through these three points into a straight line by means of a circular transformation, none of the three points being transformed to infinity. Then C is transformed into a curve that is evidently not convex.

§ 4. It is ordinarily not difficult to determine what curves are transformed into convex curves by a *single* transformation instead of a *group* of transformations. Aside from regularity, the condition for a curve C is merely that C shall not be cut by any curve l in more than two points not lying on an arc of l that is part of C, where l is any curve corresponding under the transformation to a straight line. A necessary and sufficient condition that a regular curve be transformed by an inversion into a convex curve is that no three points of it distinct from the center of inversion be concyclic with the center of inversion, unless these three points lie on an arc of the curve which is also an arc of a circle through the center of inversion.

§ 5. We shall now generalize the preceding results to three-dimensional space. If we make the usual convention of a plane at infinity, then the definition of a convex point set, the theorem of Darboux, and the proof of Theorem I hold without change for space. Hence we have

THEOREM IV. *A necessary and sufficient condition that a one-to-one point transformation of space transform every convex point set into a convex point set is that it be a collineation.*

§ 6. We shall now prove the space analogue of Theorem II:

THEOREM V. *A necessary and sufficient condition that a one-to-one point transformation of space transform every convex surface into a convex surface is that it be a collineation.*

A surface, whether closed or not, is said to be *convex* when and only when it is regular and no three points of the surface are collinear unless they lie on a line segment every point of which is a point of the surface. The sufficiency of the condition of Theorem V is therefore evident. We proceed to demonstrate its necessity.

If a one-to-one point transformation T of space transforms every convex surface into a convex surface, then it transforms every plane into a plane or a portion of a plane. For suppose a plane π is transformed by T into a surface π' not a plane nor a portion of a plane. We can cut π' by a line λ' in two points M', N' which do not lie on a segment of λ' which lies wholly in π'. Choose a point P' on λ' but not on π'. Denote by M, N, P the points which are transformed by T into M', N', P' respectively. Let σ be any plane through P intersecting π in a line ν so that ν does not separate M from N. Then that half of σ which contains P and

is bounded by ν together with the half of π which is bounded by ν and contains M and N, forms a convex surface. This convex surface is transformed by T into a surface which is not convex.

We have proved that T transforms every plane into a plane or portion of a plane. Then T transforms every line into a line or portion of a line, transforms collinear points into collinear points, and hence T is a collineation.

§ 7. Theorem III for the plane is analogous to the following theorem for space:

THEOREM VI. *A necessary and sufficient condition that a convex surface be transformed into a convex surface by every spherical transformation is that one of these transformations transform it into a plane or half-plane.*

An alternative statement of the condition is that the surface be a sphere, spherical zone bounded by a single circle, plane, half-plane, or circular disk.

Any three points of the surface determine a circle, and this circle or an arc of it containing the three points must lie entirely in the surface. Otherwise by an inversion in space we should be able to transform a fourth point of that circle to infinity; the circle would be transformed into a straight line and the surface into a surface not convex. We now transform to infinity one point of the surface, a boundary point if the surface is not closed. Every line two finite points of which lie in the surface must have in the surface an entire segment containing these two points. Then at least part of the surface is a plane π or a portion of a plane π. Moreover no finite point P outside of π can be a point of the surface, for then our surface would have to contain all points of a segment connecting P with any point of that part of the surface which lies in the plane π.

If our surface is not an entire plane, its boundary must be a straight line. Otherwise we can cut that boundary by a straight line in two finite points and an infinite point, which three points are readily inverted into three finite points on a line yet belonging to no segment of that line entirely lying in the surface corresponding. This completes the proof.*

* In § 7 there is practically all the material needed to prove the theorem:

If every plane section of a surface is a circle or an arc of a circle, the surface is either a sphere or a spherical zone bounded by a single circle.

For three points of the surface determine a plane and a circle in that plane, so any three points of the surface lie on a circle or circular arc entirely in the surface.

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